DIMENSIONAL ANALYSIS AND SIMILITUDE

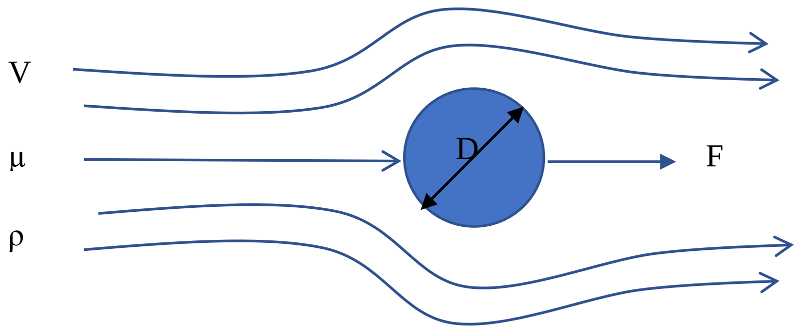
Many real fluid flow problems can be solved, at best, only approximately by using analytical or numerical methods. Therefore, experiments play a crucial role in verifying these approximate solutions.

Solutions of real problems usually involve a combination of analysis and experimental work. First, the real physical flow situation is approximated with a mathematical model that is simple enough to yield a solution. Then experimental measurements are made the check the analytical results. Based on the measurements, refinements are made in the analysis. The experimental results are an essential link in this iterative process.

The experimental work in the laboratory is both time consuming and expensive. The obvious goal is to obtain the most information from the fewest experiments.

The dimensional analysis is an important tool that often helps in achieving this goal. Dimensional analysis is packaging or compacting technique used to reduce the complexity of experimental programs and at the same time increase the generality of experimental information.

Consider the drag force on a stationary smooth sphere immersed in a uniform stream. What experiments must be conducted to determine the drag force on the sphere?



We would expect the drag force, F, depend on diameter of the sphere, D, the fluid velocity, V, fluid viscosity, μ and the fluid density ρ. That is,

$$F = f(D, V, \rho, \mu)$$

Let us imagine a series of experiments to determine the dependence of F on the variables D, V, ρ and μ. To obtain a curve of F versus V for fixed values of ρ, μ and D, we might need tests at 10 values of V. To explore the diameter effect, each test would be repeat for spheres of ten different diameters. If the procedure were repeated for 10 values of ρ and μ in turn, simple arithmetic shows that 10^4 separate test would be needed. Also we would have to find 100 different fluids. Because we need 10 different ρ's and 10 different μ's. Assuming each test takes ½ hour and we work 8 hours per day, the testing will require 2.5 years to complete.
Dimensional analysis comes to rescue. If we apply dimensional analysis, it reduces to the equivalent form.

$$\frac{F}{\rho V^2 D^2} = f_1\left(\frac{\rho V D}{\mu}\right)$$

The form of function still must be determined experimentally. However, rather than needing to conduct 10^4 experiments, we would establish the nature of function as accurately with only 10 tests.

BUCKINGHAM PI THEOREM

The dimensional analysis is based on the Buckingham Pi theorem. Suppose that in a physical problem, the dependent variable q_1 is a function of n-1 independent variables q_2, q_3, ….., q_n. Then the relationship among these variables may be expressed in the functional form as

$$q_1 = f(q_2, q_3, \dots, q_n)$$

Mathematically, we can express the functional relationship in the equivalent form.

$$g(q_1, q_2, q_3, \dots, q_n) = 0$$

Where *g* is an unspecified function, and it is different from the function *f*. For the drag on sphere we wrote the symbolic equation

$$F = f(D, V, \rho, \mu)$$

We could just as well have written

$$g(F, D, V, \rho, \mu) = 0$$

The Buckingham Pi theorem states that, the *n* parameters may be grouped into *n-m* independent dimensionless ratios, or π parameters, expressible in functional form by

$$G(\pi_1, \pi_2, \dots, \pi_{n-m}) = 0$$

or

$$\pi_1 = G_1(\pi_2, \pi_3, \dots, \pi_{n-m})$$

The number m is usually, but not always, equal to the minimum number of independent dimensions required to specify the dimensions of all the parameters, q_1, q_2, ….., q_n.

DETERMING THE Π GROUPS

To determine the π parameters, the steps listed below should be followed.

Step 1

Select all the parameters that affect a given flow phenomenon and write the functional relationship in the form

$$q_1 = f(q_2, q_3, \dots, q_n)$$

or

$$g(q_1, q_2, \dots, q_n) = 0$$

If all the pertinent parameters are not included, a relation may be obtained, but it will not give the complete story. If parameters that actually have no effect on the physical phenomenon are included, either the process of dimensional analysis will show that these do not enter the relation sought, or experiments will indicate that one or more nondimensional groups are irrelevant.

Step 2

List the dimensions of all parameters in terms of the primary dimensions which are the mass, M, the length, L, and the time, t (MLt), or the force, F, the length, L, and the time, t (FLt). Let "r" be the number of primary dimensions.

Step 3

Select a number of repeating parameters, equal to the number of primary dimensions, r, and including all the primary dimensions. As long as, the repeating parameter may appear in all of the nondimensional groups that are obtained, then do not include the dependent parameter among those selected in this step.

Step 4

Set up dimensional equation, combining the parameters selected in step 3 with each of the remaining parameters in turn, to form dimensionless groups. (There will be n-m equations). Solve the dimensional equation to obtain the (n-m) dimensionless groups.

Step 5

Check to see that each group obtained is dimensionless.

Example: The drag force, ***F***, on a smooth sphere, which is moving comparatively slowly through a viscous fluid, depends on the relative velocity, ***V***, the sphere diameter, ***D***, the fluid density, $\boldsymbol{\rho}$, and the fluid viscosity, $\boldsymbol{\mu}$. Obtain a set of dimensionless groups that can be used to correlate experimental data.

Solution:

Step 1	F	V	D	ρ	μ	n = 5 parameters
Step 2	$\frac{ML}{t^2}$	$\frac{L}{t}$	L	$\frac{M}{L^3}$	$\frac{M}{Lt}$	r = 3 primary dimensions

Step 3 Select repeating parameters *ρ, V, D*

Step 4 Then, *n-m = 2 dimensionless groups* will result. Setting up dimensional equations, we obtain,

$$\Pi_1 = \rho^a V^b D^c F = \left[\left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^b (L)^c \left(\frac{ML}{t^2}\right) \right] = \left[M^0 L^0 t^0\right]$$

Equating the exponents of M, L, and t results in

$$\left.\begin{array}{lll} M: & a+1=0 & a=-1 \\ L: & -3a+b+c+1=0 & c=-2 \\ t: & -b-2=0 & b=-2 \end{array}\right\} \text{Therefore, } \Pi_1 = \frac{F}{\rho V^2 D^2}$$

Similarly,

$$\Pi_2 = \rho^d V^e D^f \mu = \left[\left(\frac{M}{L^3}\right)^d \left(\frac{L}{t}\right)^e (L)^f \left(\frac{M}{Lt}\right) \right] = \left[M^0 L^0 t^0\right]$$

$$\left.\begin{array}{lll} M: & d+1=0 & d=-1 \\ L: & -3d+e+f-1=0 & f=-1 \\ t: & -e-1=0 & e=-1 \end{array}\right\} \text{Therefore, } \Pi_2 = \frac{\mu}{\rho V D}$$

Step 5: Check using F, L, t dimensions

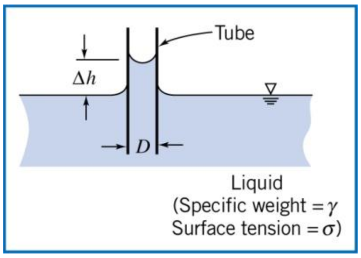
$$\Pi_1 = \frac{F}{\rho V^2 D^2} = \left[F \left(\frac{L^4}{Ft^2}\right) \left(\frac{t}{L}\right)^2 \left(\frac{1}{L^2}\right) \right] = [1]$$

$$\Pi_2 = \frac{\mu}{\rho V D} = \left[\left(\frac{Ft}{L^2}\right) \left(\frac{L^4}{Ft^2}\right) \left(\frac{t}{L}\right) \left(\frac{1}{L}\right) \right] = [1]$$

The functional relationship is

$$\Pi_1 = f(\Pi_2) \quad \text{or} \quad \frac{F}{\rho V^2 D^2} = f\left(\frac{\mu}{\rho V D}\right)$$

Example: When a small tube is dipped into a pool liquid, surface tension causes a meniscus to form at the free surface, which is elevated or depressed depending on the contact angle at the liquid-solid-gas interface. Experiments indicate that the magnitude of the capillary effect, $\boldsymbol{\Delta h}$, is a function of the tube diameter, $\boldsymbol{D}$, liquid specific weight, $\boldsymbol{\gamma}$, and surface tension, $\boldsymbol{\sigma}$. Determine the number of independent $\boldsymbol{\pi}$ parameters that can be formed and obtain a set.



Solution:

Given: $\Delta h = f(D, \gamma, \sigma)$

Find: Determine the number of independent $\boldsymbol{\pi}$ parameters and obtain a set of $\boldsymbol{\pi}$ parameters.

Step 1 Δh D γ σ $n = 4$ parameters

Step 2 Choose primary dimensions, use both *M, L, t* and *F, L, t* dimensions to illustrate the problem in determining *m.*

a) M, L, t

Δh	D	γ	σ
L	L	$\dfrac{M}{L^2t^2}$	$\dfrac{M}{t^2}$

r = 3 primary dimensions

b) F, L, t

Δh	D	γ	σ
L	L	$\dfrac{F}{L^3}$	$\dfrac{F}{L}$

r = 2 primary dimensions

Thus for each primary set of dimensions we ask, "Is m equal to r?" Let us check each dimensional matrix to find out. The dimensional matrices are,

	Δh	D	γ	σ
M	0	0	1	1
L	1	1	-2	0
t	0	0	-2	-2

	Δh	D	γ	σ
F	0	0	1	1
L	1	1	-3	-1

The rank of a matrix is equal to the order of its largest nonzero determinant.

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & -2 & 0 \\ 0 & -2 & -2 \end{vmatrix} = 0-(1)(-2)+(1)(-2)$$

$$=0$$

$$\begin{vmatrix} -2 & 0 \\ -2 & -2 \end{vmatrix} = 4 \neq 0$$

$\therefore$ m = 2

m $\neq$ r

$$\begin{vmatrix} 1 & 1 \\ -3 & -1 \end{vmatrix} = -1+3= 2 \neq 0$$

$\therefore$ m = 2

m = r

*Alternatively, you may use reduced row echelon form of the matrix to determine the rank of the matrix. The number of nonzero rows of the reduced row echelon matrix give the rank of that matrix.

Step 3 — $m = 2$. Choose ***D***, γ as repeating parameters.

Step 4 — *n-m = 2 dimensionless groups* will result

$$\Pi_1 = D^a \gamma^b \Delta h$$
$$= \left[(L)^a \left(\frac{M}{L^2 t^2} \right)^b (L) \right] = \left[M^0 L^0 t^0 \right]$$

$$\left.\begin{array}{ll} M: & b + 0 = 0 \\ L: & a - 2b + 1 = 0 \\ t: & -2b - 0 = 0 \end{array}\right\} \begin{array}{l} b = 0 \\ a = -1 \end{array}$$

$$\text{Therefore, } \Pi_1 = \frac{\Delta h}{D}$$

$$\Pi_2 = D^c \gamma^d \sigma$$
$$= \left[(L)^c \left(\frac{M}{L^2 t^2} \right)^d \left(\frac{M}{t^2} \right) \right] = \left[M^0 L^0 t^0 \right]$$

$$\left.\begin{array}{ll} M: & d + 1 = 0 \\ L: & c - 2d = 0 \\ t: & -2d - 2 = 0 \end{array}\right\} \begin{array}{l} d = -1 \\ c = -2 \end{array}$$

$$\text{Therefore, } \Pi_2 = \frac{\sigma}{D^2 \gamma}$$

Step 5 — Check using *F, L, t* dimensions

$$\Pi_1 = \frac{\Delta h}{D} = \left[\frac{L}{L} \right] = [1]$$

$$\Pi_2 = \frac{\sigma}{D^2 \gamma} = \left[\frac{F}{L} \frac{1}{L^2} \frac{L^3}{F} \right] = [1]$$

$m = 2$. Choose ***D***, γ as repeating parameters.

n-m = 2 dimensionless groups will result

$$\Pi_1 = D^e \gamma^f \Delta h$$
$$= \left[(L)^e \left(\frac{F}{L^3} \right)^f (L) \right] = \left[F^0 L^0 t^0 \right]$$

$$\left.\begin{array}{ll} F: & f = 0 \\ L: & e - 3f + 1 = 0 \end{array}\right\} \begin{array}{l} f = 0 \\ e = -1 \end{array}$$

$$\text{Therefore, } \Pi_1 = \frac{\Delta h}{D}$$

$$\Pi_2 = D^g \gamma^h \sigma$$
$$= \left[(L)^g \left(\frac{F}{L^3} \right)^h \left(\frac{F}{L} \right) \right] = \left[F^0 L^0 t^0 \right]$$

$$\left.\begin{array}{ll} F: & h + 1 = 0 \\ L: & g - 3h - 1 = 0 \end{array}\right\} \begin{array}{l} h = -1 \\ g = -2 \end{array}$$

$$\text{Therefore, } \Pi_2 = \frac{\sigma}{D^2 \gamma}$$

Check using *M, L, t* dimensions

$$\Pi_1 = \frac{\Delta h}{D} = \left[\frac{L}{L} \right] = [1]$$

$$\Pi_2 = \frac{\sigma}{D^2 \gamma} = \left[\frac{M}{t^2} \frac{1}{L^2} \frac{L^2 t^2}{M} \right] = [1]$$

Therefore, both systems of dimensions yield the same dimensionless Π parameters. The functional relationship is

$$\Pi_1 = f(\Pi_2) \quad \text{or} \quad \frac{\Delta h}{D} = f\left(\frac{\sigma}{D^2 \gamma} \right)$$

DIMENSIONLESS GROUPS OF SIGNIFICANCE IN FLUID MECHANICS

There are several hundred dimensionless groups in engineering. Following tradition, each such group has been given the name of a prominent scientist or engineer, usually the one who pioneered its use.

Forces encountered in the flowing fluids include those due to inertia, viscosity, pressure, gravity, surface tension, and compressibility. The ratio of any two forces will be dimensionless. We can estimate typical magnitudes of some of these forces in a flow:

$$\text{Inertia force} = m\vec{a} = m\frac{D\vec{V}}{Dt} = m\left[u\frac{\partial u}{\partial x}\ldots\right] \propto \rho L^3 V\frac{V}{L} = \rho L^2 V^2$$

$$\text{Viscous force} = \tau A = \mu\frac{du}{dy}A \propto \mu\frac{V}{L}L^2 = \mu VL$$

$$\text{Pressure force} = (\Delta p)A \propto \Delta p L^2$$

$$\text{Gravity force} = mg \propto g\rho L^3$$

$$\text{Surface tension force} = \sigma L$$

$$\text{Compressibility force} = E_v A \propto E_v L^2$$

$$E_v \equiv \frac{dp}{d\rho/\rho}$$

Inertia forces are important in most fluid mechanics problems. The ratio of the inertia force to each of other forces listed above leads to five fundamental groups encountered in fluid mechanics.

The Reynolds number is the ratio of inertia forces to the viscous forces, and it is named after Osbourne Reynolds (1842 - 1912).

$$\text{Reynolds number} = \text{Re} = \frac{\text{Inertia force}}{\text{Viscous force}} = \frac{\rho V^2 L^2}{\mu VL} = \frac{\rho VL}{\mu} = \frac{VL}{\nu}$$

$$\text{Euler number} = \text{Eu} = f\left(\frac{\text{Pressure force}}{\text{Inertia force}}\right) = \frac{\Delta p L^2}{\frac{1}{2}\rho V^2 L^2} = \frac{\Delta p}{\frac{1}{2}\rho V^2}$$

where Δp is the pressure difference between local pressure and the freestream pressure.

$$\Delta p = p - p_\infty$$

$$\text{Froude number} = \text{Fr} = f\left(\frac{\text{Inertia force}}{\text{Gravity force}}\right) = \left(\frac{\rho V^2 L^2}{\rho g L^3}\right)^{1/2} = \frac{VL}{\sqrt{gL}}$$

$$\text{Weber number} = \text{We} = f\left(\frac{\text{Inertia force}}{\text{Surface tension force}}\right) = \frac{\rho V^2 L^2}{\sigma L} = \frac{\rho V^2 L^2}{\sigma L}$$

$$\text{Mach number} = \text{M} = f\left(\frac{\text{Inertia force}}{\text{Compressibility force}}\right) = \left(\frac{\rho V^2 L^2}{E_\upsilon L^2}\right)^{1/2} = \frac{V}{\sqrt{\frac{E_\upsilon}{\rho}}} = \frac{V}{c}$$

where c is the local sonic speed.

FLOW SIMILARITY AND MODEL STUDIES

When an object, which is in original sizes, is tested in laboratory it is called **prototype**. A **model** is a scaled version of the prototype. A model which is typically smaller than its prototype is economical, since it costs little compared to its prototype. The use of the models is also practical, since environmental and flow conditions can be rigorously controlled. However, models are not always smaller than the prototype. As an example, the flow in a carburetor might be studied in a very large model.

There are three basic laws of similarity of model and prototype flows. All of them must be satisfied for obtaining complete similarity between fluid flow phenomena in a prototype and in a model. These are

a) The geometric similarity,
b) the kinematic similarity, and
c) the dynamic similarity.

Geometric Similarity: The geometric similarity requires that the model and prototype be identical in shape but differ in size. Therefore, ratios of the corresponding linear dimensions in the prototype and in the model are the same.

Kinematic Similarity: The kinematic similarity implies that the flow fields in the prototype and in the model must have geometrically similar sets of streamlines. The velocities at corresponding points are in the same direction and are related in magnitude by a constant scale factor.

Dynamic Similarity: When two flows have force distributions such that identical types of forces are parallel and are related in magnitude by a constant scale factor at all corresponding points, the flows are dynamically similar.

By using Buckingham Π theorem, we can find which dimensionless groups are important for a given flow phenomenon. To achieve dynamic similarity between geometrically similar flows, we must duplicate all of these dimensionless groups.

For example, in considering the drag force on sphere we found that

$$\frac{F}{\rho V^2 D^2} = f_1\left(\frac{\rho V D}{\mu}\right) = f_1(\text{Re})$$

Thus in considering a model flow and prototype flow about a sphere, the flows will be dynamically similar if

$$\left(\frac{\rho VD}{\mu}\right)_{\text{model}} = \left(\frac{\rho VD}{\mu}\right)_{\text{prototype}}$$

that is

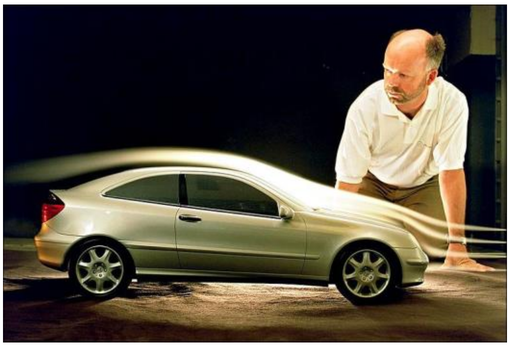
$$\text{Re}_{\text{model}} = \text{Re}_{\text{prototype}}$$

then

$$\left(\frac{F}{\rho V^2 D^2}\right)_{\text{model}} = \left(\frac{F}{\rho V^2 D^2}\right)_{\text{prototype}}$$

The results determined from the model study can be used to predict the drag on the full scale prototype.

Example: A one-tenth-scale model of a derby car, shown in the figure, is tested in a wind tunnel. The air speed in the wind tunnel is 70 m/s, the air drag on the model car is 240 N, and the air temperature and pressure are identical those expected when the prototype car is racing. Find the corresponding racing speed in still air and the drag on the car.



Solution:

The functional relation for the drag force can be found by applying Buckingham-Π theorem such that

$$\frac{F_D}{\rho V^2 L^2} = f\left(\frac{\rho VD}{\mu}\right) \qquad \text{Re} = \frac{\rho VL}{\mu}$$

and the test should be run at

$$\text{Re}_{\text{model}} = \text{Re}_{\text{prototype}}$$

$$\frac{\rho_m V_m L_m}{\mu_m} = \frac{\rho_p V_p L_p}{\mu_p}$$

to ensure dynamic similarity The problem statements show that $\rho_m = \rho_p$ and $\mu_m = \mu_p$. Then,

$$V_p = V_m \frac{L_m}{L_p}\frac{\rho_m}{\rho_p}\frac{\mu_p}{\mu_m} \qquad \Rightarrow V_p = V_m \frac{L_m}{L_p}$$

$$V_p = V_m \frac{L_m}{L_p} = 70\frac{1}{10} = 7\ \text{m/s}$$

This speed is low enough to neglect compressibility effects. At these test conditions, the model and the prototype flows are dynamically similar. Hence,

$$\left.\frac{F_D}{\rho V^2 L^2}\right)_m = \left.\frac{F_D}{\rho V^2 L^2}\right)_p$$

and

$$F_{D\,p} = F_{Dm}\left(\frac{\rho_p}{\rho_m}\right)\left(\frac{V_p^2 L_p^2}{V_m^2 L_m^2}\right) = F_{Dm}\left(\frac{V_p L_p}{V_m L_m}\right)^2$$

$$F_{D\,p} = F_{Dm}\left(\frac{V_p L_p}{V_m L_m}\right)^2 = 240\left(\frac{7}{70}\frac{10}{1}\right)^2 = 240\ \text{N}$$

Example: A jet plane travelling at a velocity of **900 m/s at 6 km altitude**, where the temperature and the pressure are **-24 °C** and **47.22 kPa**, respectively. A **one-tenth scale model** of the jet is tested in a wind tunnel in which **carbon dioxide** is flowing. The gas constant for air and carbon dioxide are **287 J/kg K** and **18.8 J/kgK**, respectively. The specific heat ratios for air and carbon dioxide are **1.4** and **1.28**, respectively. Also the absolute viscosities of the air at -24 °C and carbon dioxide at 20 °C are **1.6×10^{-5} Pa.s** and **1.47×10^{-5} Pa.s**, respectively.

Solution:

Determine
a)The required velocity in the model, and
b)The pressure required in the wind tunnel.

a) As long as the model jet plane is moving in a compressible fluid, then a free surface does not exist. Therefore, it is not necessary to concern either with the wave or surface tension effects. The Froude and the Weber numbers play no role for the dynamic similarity. In order to achieve dynamic similarity, the Reynolds numbers and Mach numbers must be equal on the model and on the prototype.

$$M_p = \frac{V_p}{c_m} = \frac{V_m}{c_m} = M_m$$

Then the velocity of the model jet plane is

$$V_m = V_p \frac{c_p}{c_m} = V_p\left(\frac{k_m R_m T_m}{k_p R_p T_p}\right)^{1/2} = 900\left(\frac{1.28\times 187.8\times 293}{1.4\times 287\times 249}\right)^{1/2} = 755.14\ \text{m/s}$$

b) The other requirement for the dynamic similarity is the equality of the Reynolds numbers

$$\mathrm{Re_p} = \frac{\rho_p V_p L_p}{\mu_p} = \frac{\rho_m V_m L_m}{\mu_m} = \mathrm{Re_m}$$

The density of air may be evaluated by using equation of state for a perfect gas

$$\rho_p = \frac{p_p}{R_p T_p} = \frac{47220}{287 \times 249} = 0.661 \ \ \frac{\mathrm{kg}}{\mathrm{m^3}}$$

Now, required density of the carbon dioxide may be evaluated as

$$\rho_m = \rho_p \frac{L_p V_p \mu_p}{L_m V_m \mu_m} = 0.661 \times 10 \times \frac{900 \times 1.47 \times 10^{-5}}{755.14 \times 1.60 \times 10^{-5}} = 7.24 \ \ \frac{\mathrm{kg}}{\mathrm{m^3}}$$

Finally, the required pressure of the carbon dioxide is

$$p_m = \rho_m R_m T_{pm} = 7.24 \times 187.7 \times 293 = 398.38 \ \ \mathrm{kPa}$$

INCOMPLETE SIMILARITY

To achieve complete dynamic similarity between geometrically similar flows all of the dimensionless numbers in prototype and in the model (that is *Re, Eu, Fr, We, M,..*) should be equal.

Fortunately, in most engineering problems, the equality of all of dimensionless groups is not necessary. Since some of forces

i. may not act
ii. may be negligible magnitude or
iii. may oppose other forces in such a way that the effect of both is reduced.

In some cases, complete dynamic similarity may not be attainable. Determining the drag force of surface ship is on example of such a situation. The viscous shear stress and surface wave resistance cause the drug. So that for complete dynamic similarity, both Reynolds and Froude numbers must be equal between model and prototype. This requires that

$$Fr_m = \frac{V_m}{(gL_m)^{\frac{1}{2}}} = Fr_p = \frac{V_p}{(gL_p)^{\frac{1}{2}}}$$

$$\frac{V_m}{V_p} = \left(\frac{L_m}{L_p}\right)^{\frac{1}{2}}$$

To ensure dynamically similar surface wave patterns.

From the Reynolds number requirement

$$Re_m = \frac{V_m L_m}{\nu_m} = Re_p = \frac{V_p L_p}{\nu_p}$$

$$\frac{\nu_m}{\nu_p} = \frac{V_m}{V_p}\frac{L_m}{L_p}$$

If we use the velocity ratio obtained from matching Froude numbers, equality of Reynolds number leads to a kinematic viscosity ratio of

$$\frac{\nu_m}{\nu_p} = \left(\frac{L_m}{L_p}\right)^{\frac{1}{2}}\left(\frac{L_m}{L_p}\right) = \left(\frac{L_m}{L_p}\right)^{\frac{3}{2}}$$

If L_m/L_p equals 1/100 (a typical length scale for ship model tests), then ν_m/ν_p must be 1/1000. Mercury, which is the only liquid, its kinematic viscosity is less than water. Thus, we cannot simultaneously match Reynolds number and Froude number in the scale-model test. Then one is forced to choose ether the Froude number similarity, or the Reynolds number similarity. For this reason, the experiments with the model are performed so that $Fr_p = Fr_m$ which results $Re_p >> Re_m$. The test results are then corrected by using the experimental data which is dependent on the Reynolds number.

Example: The drag force on a submarine, which is moving on the surface, is to be determined by a test on a model which is scaled down to **one-twentieth of the prototype**. The test is to be carried in a towing tank, where the model submarine is moved along channel of liquid. The density and the kinematic viscosity of the seawater are **1010 kg/m³** and **1.3×10⁻⁶ m²/s** respectively. The speed of the prototype is **2.6 m/s**.

a) Determine the speed at which the model should be moved in the towing tank.
b) Determine the kinematic viscosity of the liquid that should be used in the towing tank.
c) If such a liquid is not available, then the test may be carried out with seawater by neglecting the viscous effects. In this case, determine the ratio of the drag force due to the surface waves in the prototype to the drag force in the model.

a) Because of low speed of the submarine, the compressibility has no effect on the dynamic similarity, and the Mach number plays no role.
The Froude numbers for the prototype and the model may be equated to yield.

$$Fr_p = \frac{V_p}{\left(gL_p\right)^{\frac{1}{2}}} = \frac{V_m}{\left(gL_m\right)^{\frac{1}{2}}} = Fr_m$$

$$V_m = V_p\left(\frac{L_m}{L_p}\right)^{\frac{1}{2}} = 2.6\left(\frac{1}{20}\right)^{\frac{1}{2}} = 0.58\, m/s$$

b) To determine the kinetic viscosity of the liquid that should be used in the towing tank, one may equate the Reynolds number in the model and prototype.

$$Re_p = \frac{V_p L_p}{\nu_p} = \frac{V_m L_m}{\nu_m} = Re_m$$

Rearranging one may obtain

$$\nu_m = \nu_p\left(\frac{V_m}{V_p}\right)\left(\frac{L_m}{L_p}\right) = 1.3\text{x}10^{-6}\left(\frac{0.58}{2.6}\right)\left(\frac{1}{20}\right) = 1.45\text{x}10^{-8}\, m^2/s$$

c) However, one should note that a liquid with a given kinematic viscosity cannot be practically formed. Then the test in towing tank may be carried out with seawater by neglecting the viscous effects. In this case, only the equality of the Froude number is sufficient for the dynamic similarity and the drag force is only due to the surface waves.
By using Buckingham π theorem one may obtain.

$$\frac{F}{\rho V^2 L^2} = f(Re, Fr)$$

But in this case only the equality of the Froude number is sufficient, then

$$\frac{F}{\rho V^2 L^2} = f(Fr)$$

$Fr_{model}=Fr_{prototype}$

$$\left(\frac{F}{\rho V^2 L^2}\right)_{model} = \left(\frac{F}{\rho V^2 L^2}\right)_{prototype}$$

$$\frac{F_p}{F_m} = \frac{\rho_p}{\rho_m}\frac{{V_p}^2}{{V_m}^2}\frac{{L_p}^2}{{L_m}^2}$$

$$V_m = V_p\left(\frac{L_m}{L_p}\right)^{\frac{1}{2}} \quad \Rightarrow \quad \frac{V_m}{V_p} = \left(\frac{L_m}{L_p}\right)^{\frac{1}{2}}$$

$$\frac{F_p}{F_m} = \frac{L_p}{L_m}\left(\frac{L_p}{L_m}\right)^2 = \left(\frac{L_p}{L_m}\right)^3 = 20^3 = 8000$$

This result must be corrected for viscous effects.