INTRODUCTION

Fluid Mechanics in Engineering

Fluid mechanics deals with behavior of fluids at rest and in motion. Many engineering applications involve fluid in motion or stationary. Examples include home and city water supply system, transportation of oil and natural gas in pipelines, flow of blood in vessels, air flow over an aircraft, motion of a ship in water, and many others. Design and operation of all such devices require a good understanding of fluid behavior when it is stationary or in motion, and its interaction with the surface in contact.

Definition of a Fluid

Consider imaginary chunks of both a solid and a fluid. Chunks are fixed along one edge, and a shear force is applied at the opposite edge. A short time after application of the force, the solid assumes a deformed shape which can be measured by the angle ϕ_1. If we maintain this force and examine the solid at a later time, we find that deformation is exactly the same, that is $\phi_2=\phi_1$. On application of a shear force, a solid assumes a certain deformed shape and retains that shape as long as the force is applied.

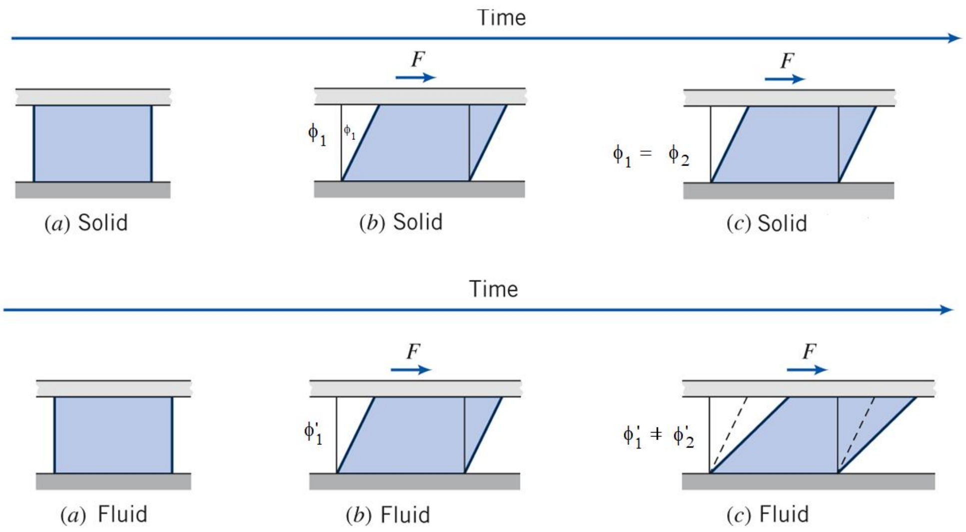


Figure 1. Solid and fluid behavior under shear stress.

Consider the response of the fluid to the applied shear force. A short time after application of the force, a fluid assumes a deformed shape, as indicated by the angle ϕ_1'. At a later time, the deformation is greater, $\phi_2'>\phi_1'$, in fact the fluid continues to deform as long as the force is applied. Thus we can define a fluid:

A **fluid** is a substance that deforms continuously under the action of applied shear force.

The process of continuous deformation is called **flowing**.

Scope of Fluid Mechanics

As pointed out above, many engineering applications involve fluids in motion or stationary. We cannot consider all these specific problems of fluid mechanics. Instead, the purpose of this course is to introduce the basic laws and associated physical concepts that provide the basis or starting point in the analysis of any problem in fluid mechanics.

Basic Equations

Analysis of any fluid mechanics problem begins, either directly or indirectly with the basic laws governing the fluid motion. The basic laws, which are applicable to any fluid, are,

1. Conservation of mass
2. Newton's second law of motion
3. Moment of momentum
4. The first law of thermodynamics
5. The second law of thermodynamics

It should be emphasized that not all basic laws are required to solve every problem. However, in some problems, it is necessary to bring into the analysis additional relations, in the form of equation of state or constitutive equations; i.e. equation of state

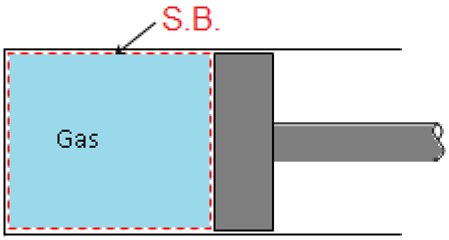
$$p = \rho RT$$

METHOD OF ANALYSIS

The first step in solving a problem is to define the system that is going to be analyzed. The basic laws can be applied to a control volume or to a system.

System and Control Volume

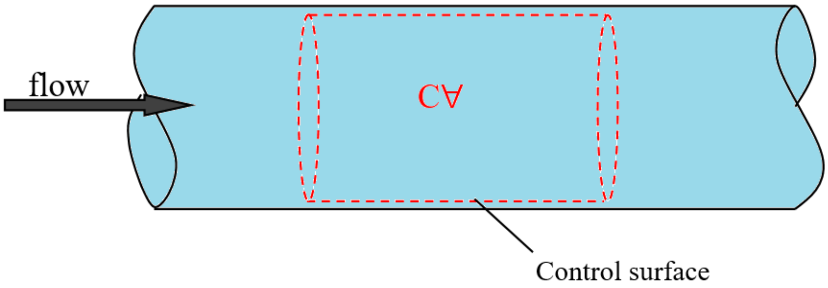
A system is defined as a fixed, identifiable quantity of mass.



The boundaries of a system may be fixed or moveable; however, there is no mass transfer across the system boundaries; i.e. the amount of mass in the system is fixed.

Control Volume

A control volume is an arbitrary volume in space through which fluid flows.



Differential vs. Integral Approach

The basic laws that we apply in fluid mechanics problems can be formulated in differential and integral forms. The solution of differential equations provides a means of determining the detailed (point by point) behavior of the basic laws.

System of Units

The SI system of units will be used. In the SI system of units

quantity	**unit**
mass	kg
length	m
time	sec
temperature	K
force	N

FUNDAMENTAL CONCEPTS

Fluid as a Continuum

All fluids are composed of molecules in constant motion. However, in most engineering applications we are interested in the average or macroscopic effects of many molecules. We thus treat a fluid as an infinitely divisible substance, a continuum, and do not concern with the behavior of individual molecule.

For continuum model to be valid, the smallest sample of the matter of practical interest must contain a large number of molecules so that meaningful averages can be calculated.

The condition for the validation of continuum approach is that distance between the molecules of the fluid should be smaller than the smallest characteristic length of the problem.

As a consequence of the continuum assumption fluid properties and flow properties can be expressed as continuous functions of position and time, i.e.

$$\rho = \rho\,(x,y,z,t)$$

$$u = u\,(x,y,z,t)$$

$$T = T\,(x,y,z,t)$$

$$p = p\,(x,y,z,t)$$

…

The value of a fluid property ar a point is defined as an average considering a volume around that point.

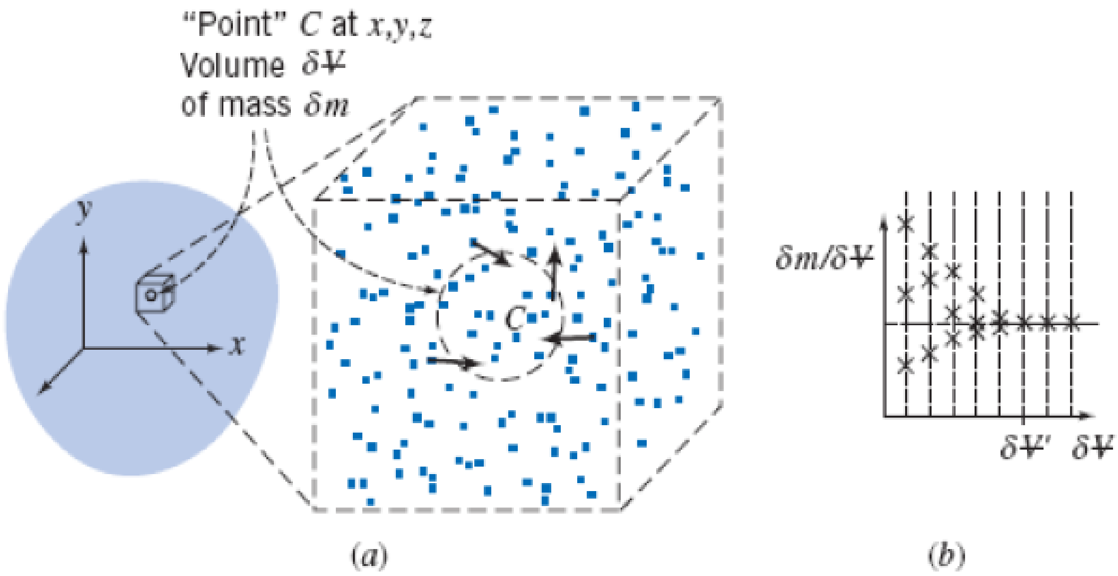
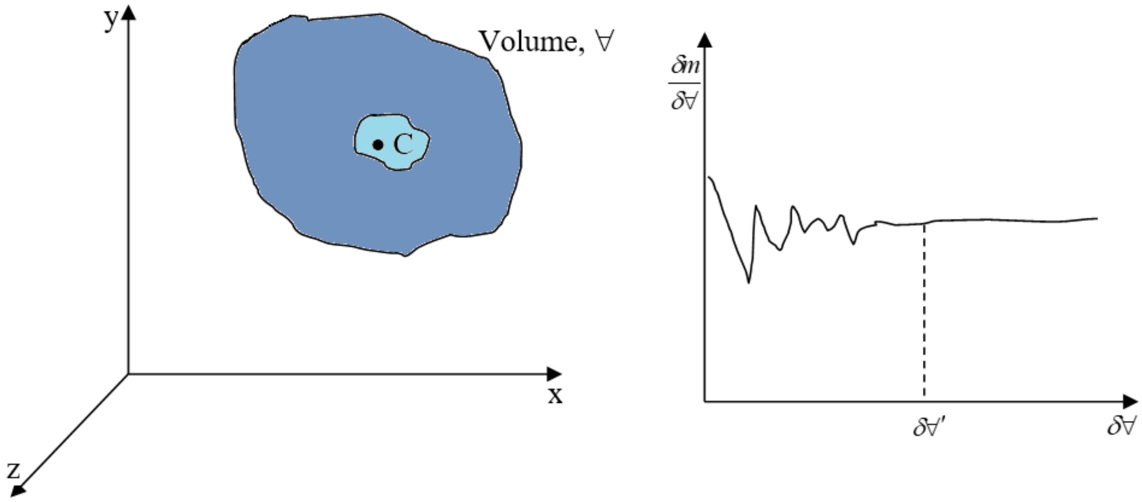


Fig. 2.1 Definition of density at a point.



VELOCITY FIELD

Continuum assumption led to description of all the fluid properties at every point in the flow domain.

The fluid velocity at a point C is defined as the velocity of the center of gravity of volume $\delta\forall'$ surrounding the point C.

The velocity at any point in the flow field is a function of space and time, i.e.

$$\vec{V} = \vec{V}(x, y, z, t)$$

Velocity vector $\vec{V}$, can be written in terms of scalar components

$\vec{V} = u\vec{\imath} + v\vec{j} + w\vec{k}$, $\quad$ *u* is x-component of velocity

v is y-component of velocity

w is z-component of velocity

Steady Flow

If properties at each point in a flow do not change with time, the flow is called steady. Mathematically for any property η

$$\frac{\partial \eta}{\partial t} = 0$$

or

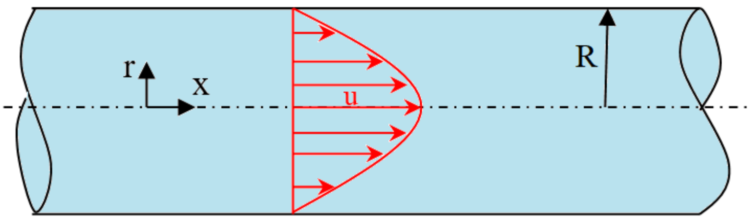
$$\frac{\partial \rho}{\partial t} = 0 \qquad \Rightarrow \qquad \rho = \rho(x, y, z)$$

$$\frac{\partial \vec{V}}{\partial t} = 0 \qquad \Rightarrow \qquad \vec{V} = \vec{V}(x, y, z)$$

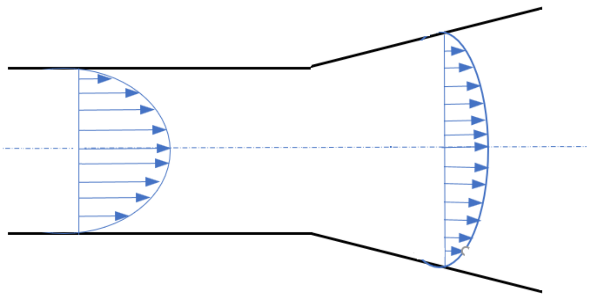
ONE- TWO- AND THREE-DIMENSIONAL FLOWS

A flow is classified as one-, two-, or three-dimensional depending on the number of space coordinates required to specify the velocity field.

Example:



$u = u_{max}\left[1-\left(\frac{r}{R}\right)^2\right]$ velocity depends on only r, hence the flow is one-dimensional.

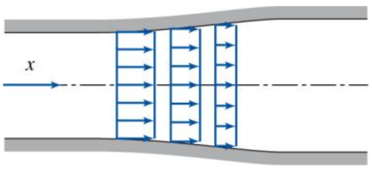


Velocity changes with x and y coordinates, hence the flow is two dimensional.

Give example about three dimensional flows.

Uniform Flow

To simplify the analysis, sometimes velocity at a cross-section is assumed to be constant. If velocity at a given cross section is assumed to be uniform, flow is called uniform flow.



at all points at given cross section velocity is same.

Timelines, Pathlines, Streaklines, and Streamlines

Timelines, pathlines, streaklines and streamlines provide a visual representation of a flow field.

Timeline:

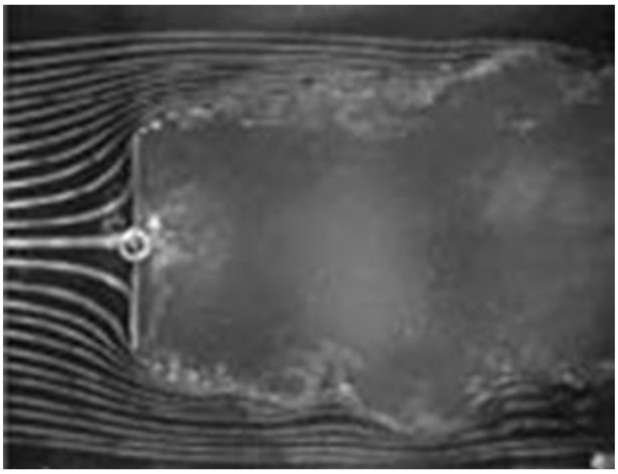
If a number of adjacent fluid particles in a flow field are marked at a given instant, they form a line in the fluid at that instant, this line is called a **timeline**. Observation of the timeline at a later instant may provide information about the flow field.

Pathline:

A **pathline** is the path or trajectory traced out by a moving fluid particle. A pathline may be obtained by following a fluid particle (i.e. by use of dye) in the flow field.

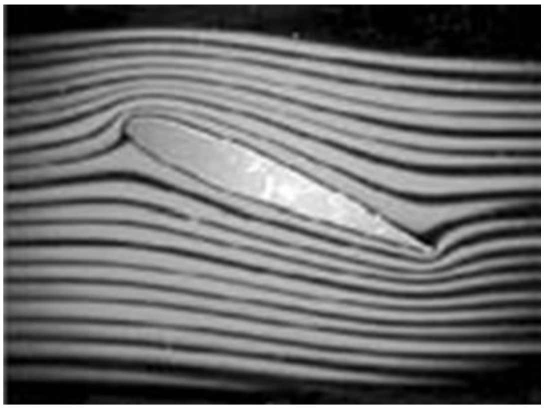
Streakline:

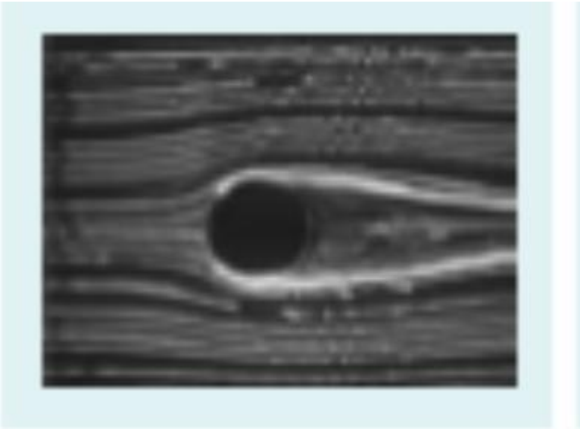
A line joining the fluid particles that pass through the same point in the flow field is called the **streakline**.

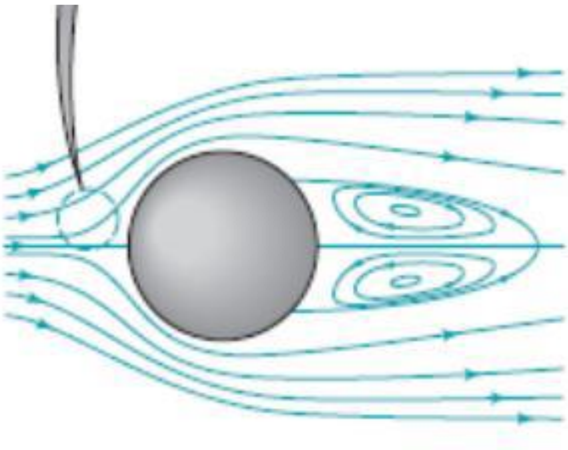


Streamline:

Streamlines are lines drawn in the flow field so that at given instant they are trangenbt to direction of flow at every point in the flow field. Streamlines are tangent to the velocity vector at every point in the flow field.







In steady flow, pathlines, streaklines, and streamlines are identical lines in the flow field.

Example: A velocity given by $\vec{V} = ax\vec{i} - ay\vec{j}$, the units of velocity are m/s; and x and y are given in meters; $a=0.1\ sec^{-1}$.

a) Determine the equation for the streamline passing through the point $(x_0, y_0, 0)=(2, 8, 0)$
b) Determine the velocity of a particle at the point (2, 8, 0)
c) If the particle passing through the point $(x_0, y_0, 0)$ is marked at time $t_0=0$, determine the location of the particle at time t=20 sec.
d) What is the velocity of the particle at t=20 sec.
e) Show that the equation of the pathline is the same as the equation of the streamline.

a) Equation of streamline through point (2, 8, 0)

Streamlines are tangent to the flow direction (velocity vector). Hence,

$$\left.\frac{dy}{dx}\right|_{streamline} = \frac{v}{u} = \frac{-ay}{ax} = -\frac{y}{x}$$

separating variables and integrating

$$\int \frac{dy}{y} = -\int \frac{dx}{x} \quad \Rightarrow \ln y = -\ln x + C_1 \qquad \text{or } xy=C$$

For the streamline passing through point (2, 8, 0), the constant C

$C=2\times 8=16$

and the equation of the streamline through point (2, 8,0)

$xy=16\ m^2$

b) The velocity field is $\vec{V}=ax\vec{i}-ay\vec{j}$

at point (2, 8, 0) is $\vec{V}=0.2\vec{i}-0.8\vec{j}$

c) $u_p=\dfrac{dx}{dt}=ax \Rightarrow \displaystyle\int_{x_0}^{x}\frac{dx}{x}=\int_0^t a\,dt \Rightarrow \ln\frac{x}{x_0}=at \Rightarrow x=x_0e^{at}$

$v_p=\dfrac{dy}{dt}=-ay \Rightarrow \displaystyle\int_{y_0}^{y}\frac{dy}{y}=\int_0^t -a\,dt \Rightarrow \ln\frac{y}{y_0}=-at \Rightarrow y=y_0e^{-at}$

at t=20 sec., $x=2e^{(0.1\times 20)}=14.8$ m.

$y=8\ e^{-(0.1\times 20)}=1.08$ m.

$\therefore$ at t=20 sec., particle is at point (14.8, 1.08, 0) m.

d) t=20 sec particle is at point (14.8, 1.08, 0)

$\therefore$ velocity at this point $\vec{V}=0.2(14.8\vec{i}-1.08\vec{j})=1.48\vec{i}-0.108\vec{j}$

e) To determine equation of the pathline, we use the parametric equations $x=x_0e^{at}$ and $y=y_0e^{-at}$

Solving for e^{at},

$$e^{at}=\frac{y_0}{y}=\frac{x}{x_0}$$

$\therefore xy=x_0y_0=16\ m^2$ Equation of pathline for particle passing through (x_0, y_0, 0).

Streamline passing through the point (x_0, y_0, 0) can also found as $xy=16\ m^2$

$\therefore$ Pathline and the streamline passing through (x_0, y_0, 0) are the same for steady flow.

STRESS FIELD

Forces acting on a fluid element

- Surface forces
- Body forces

Surface forces include all forces acting on the boundaries of a medium through direct contact. Forces developed without physical contact, and distributed over the volume of the fluid are called body forces.

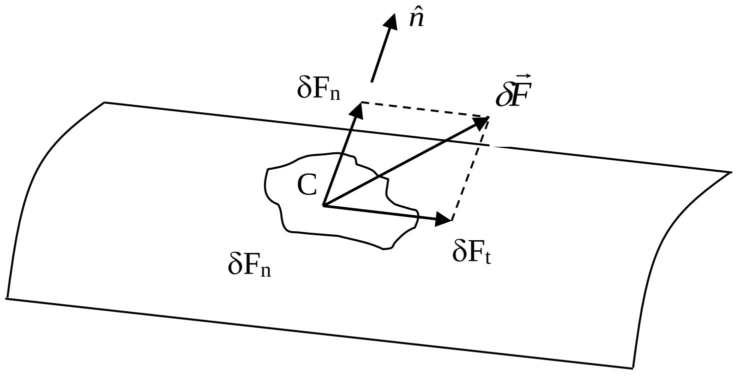
Gravitational and electromagnetic forces are body forces.

Gravitational body force acting on a fluid element of volume d∀ is $\vec{g}\rho d\forall$

and gravitational body force acting on per unit volume of a fluid element is $\rho\vec{g}$.

The concept of stress field provides a convenient means to describe forces acting on boundaries of a fluid medium and transmitted through the medium.

Consider an area δA around point C in a continuum. The force acting $\delta\vec{F}$ acting on $\delta\vec{A}$ can be resolved into two components, one normal and the other tangential to the area



δF_n : normal component

δF_n : tangential component

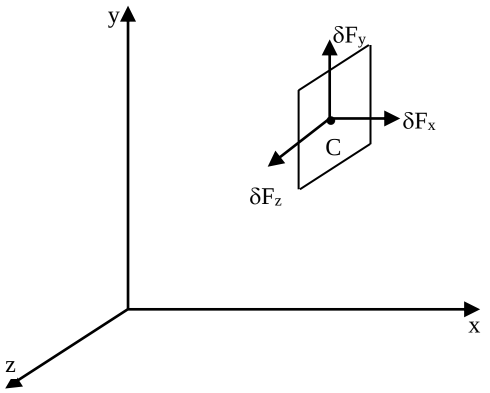
$\hat{n}$: normal unit vector

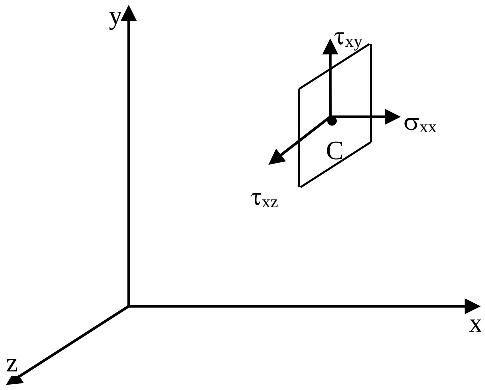
Normal stress σ_n and shear stress τ_n are defined as

$$\sigma_n = \lim_{\delta A_n \to 0} \frac{\delta F_n}{\delta A_n} \qquad \text{and} \qquad \tau_n = \lim_{\delta A_n \to 0} \frac{\delta F_t}{\delta A_n}$$

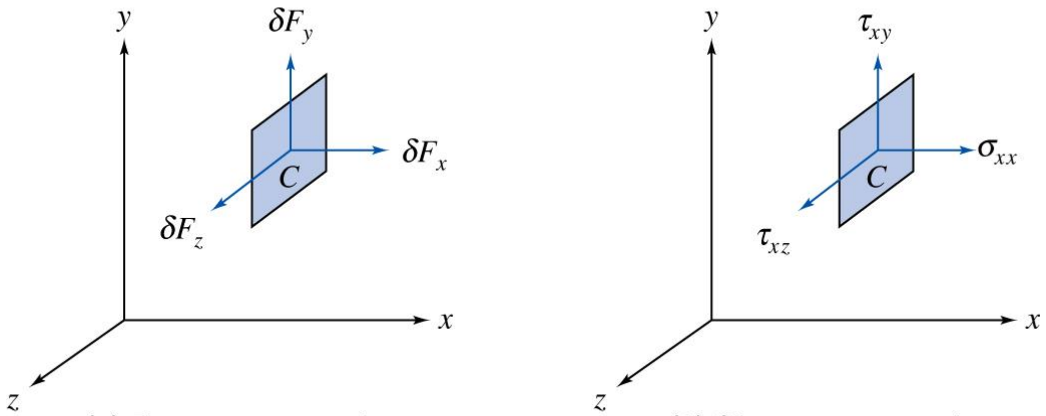
Note: subscript, n, indicates that the stress are associated with a particular surface $\delta\vec{A}$ through point C.

Note that a point C in a continuum different surfaces can be drawn. However, for purpose of analysis, we usually reference the area to some coordinate system. In rectangular coordinate system, we might consider the stress acting on planes whose outward drawn normals are in x,y or z-directions.





Force components on element of area δA_x Stress components on element of area δA_x



(*a*) Force components (*b*) Stress components

Stress components shown in above figure is defined as

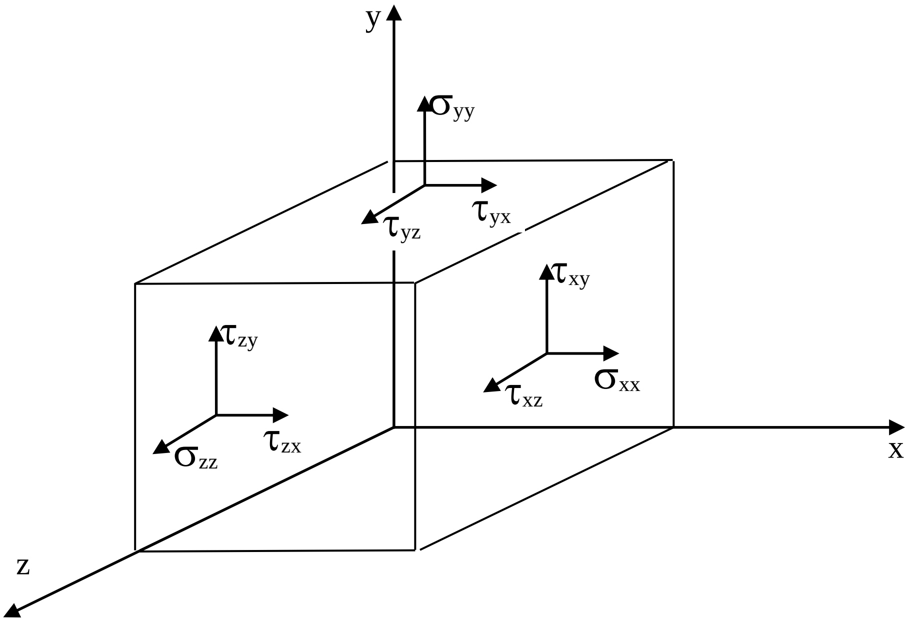
$$\sigma_{xx} = \lim_{\delta A_x \to 0} \frac{\delta F_n}{\delta A_x} \ , \ \tau_{xy} = \lim_{\delta A_x \to 0} \frac{\delta F_y}{\delta A_x} \ , \ \tau_{xz} = \lim_{\delta A_x \to 0} \frac{\delta F_z}{\delta A_x}$$

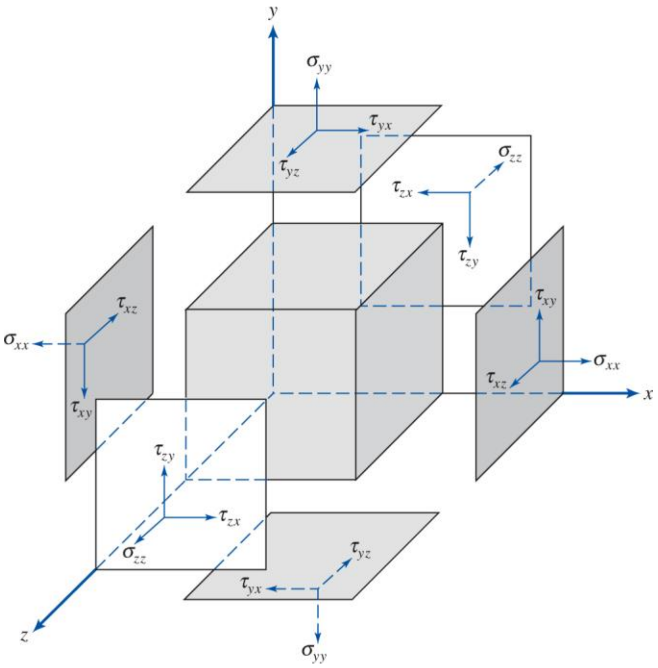
We have used a double subscript notation to label the stresses.

$\tau_{i,j}$ i: indicates plane on which stress acts (plane perpendicular to axis i)

j: direction in which stress acts

Consideration of an area element, δA_y, would lead to the definition of stresses σ_{yy}, τ_{yx}, τ_{yz}, and use of area element δA_z would similarly lead to the definitions of σ_{zz}, τ_{zx}, τ_{zy}.





An infinite number of planes can be passed through point C, resulting an infinite number of stresses associated with that point. Fortunately state of stress at a point can be described completely by specifying the stresses acting on three mutually perpendicular planes through the point. Hence, stress at a point is specified by the nine components.

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

The planes are named in terms of the coordinate axes. The planes are named and denoted as positive or negative according to the direction of the outward drawn normal to the plane. Thus, the top plane for example is a positive y-plane and the back plane is a negative z-plane.

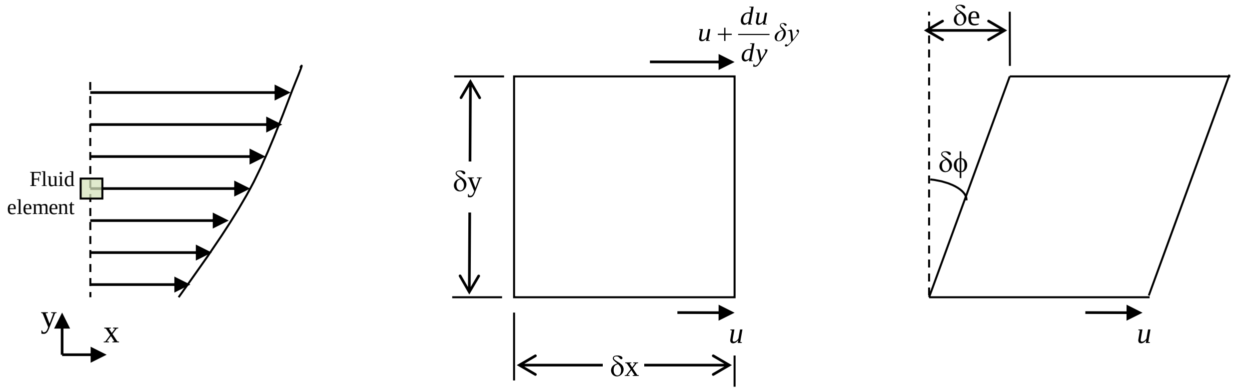
It is also necessary to adopt a sign convention for stress. A stress component is considered positive when the direction of the stress component and the plane on which it acts are both positive or both negative. In other words, a shear stress on positive y-plane in positive x-direction or shear stress on negative y-plane in negative x-direction.

Thus, τ_{yx}**=2.4 N/m²** represents a shear stress **on positive y-plane** in **positive x-direction** or shear stress on **negative y-plane** in **negative x-direction**.

VISCOSITY

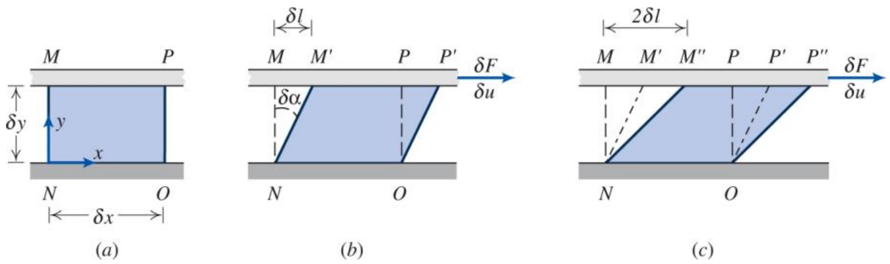
We have learned that a fluid is a substance that undergoes continuous deformation when subjected to a shear stress. This shear stress is a function of rate of deformation. For many common the shear stress is proportional to the rate of deformation. The constant of proportionality, called *viscosity*, is a fluid property.

To develop the defining equation for viscosity, we consider a flow in x-y plane in which x-direction velocity varies with y.



Fluid element at time t

Fluid element at time t+δt



Consider the fluid element in the figure. The top of the fluid element moves faster than the bottom, so in time fluid element deform.

We measure shear deformation by the angle δφ, which can be related to the fluid velocity.

$$\left.\begin{array}{l} \delta e = \left(u + \dfrac{du}{dy}\delta y\right)\delta t - u\delta t = \dfrac{du}{dy}\delta y \delta t \\ also \\ \tan\delta\phi \cong \delta\phi = \dfrac{\delta e}{\delta y} \Rightarrow \delta e = \delta\phi\delta y \end{array}\right\} \Rightarrow \delta\phi\delta y = \frac{du}{dy}\delta y \delta t$$

$$\frac{\delta\phi}{\delta t} = \frac{du}{dy} \qquad Shear\,deformation\,rate$$

Hence shear stress

$$\tau_{yx} \propto \frac{\delta\phi}{\delta\phi}$$

$$or$$

$$\tau_{yx} \propto \frac{du}{dy}$$

Newtonian Fluid

Fluid in which constant of proportionality in above expression is equal to the viscosity called *Newtonian fluid*.

Newton's law of viscosity:

$$\tau_{yx} = \mu \frac{du}{dy}$$ μ: dynamic viscosity (absolute viscosity)

Unit of μ

$$\left.\begin{array}{l} \tau : \dfrac{F}{L^2} \\ \dfrac{d}{dy} := \dfrac{1}{t} \end{array}\right\} \Rightarrow \mu : \frac{Ft}{L^2}$$

In metric system $poise \equiv \dfrac{g}{cm.\sec}$

$$1\, poise = 0.1 \frac{kg}{m \sec}$$

In SI $\mu : \dfrac{N \sec}{m^2}$

$$\mu : \frac{kg\, m}{\sec}$$

Kinematic Viscosity ν

$$\nu = \frac{\mu}{\rho}, \quad \left[\frac{L^2}{t}\right], \left[\frac{m^2}{\sec}\right]$$

In metric system $stoke \equiv \dfrac{cm^2}{\sec}$

$$1\, poise = 0.0001 \frac{m^2}{\sec}$$

Non-Newtonian Fluid

Not all fluids follow the Newton's law of viscosity (stress-strain relation). Such fluids are called non-Newtonain. Some fluids such as ketchup, are 'shear-thinning'; that is the coefficient of resistance decreases with increasing strain rate (it all comes out of the bottle at once). Others, such as a mixture of sand and water 'shear-thickening'. Some fluids do not begin to flow until a finite stress been applied (toothpaste).

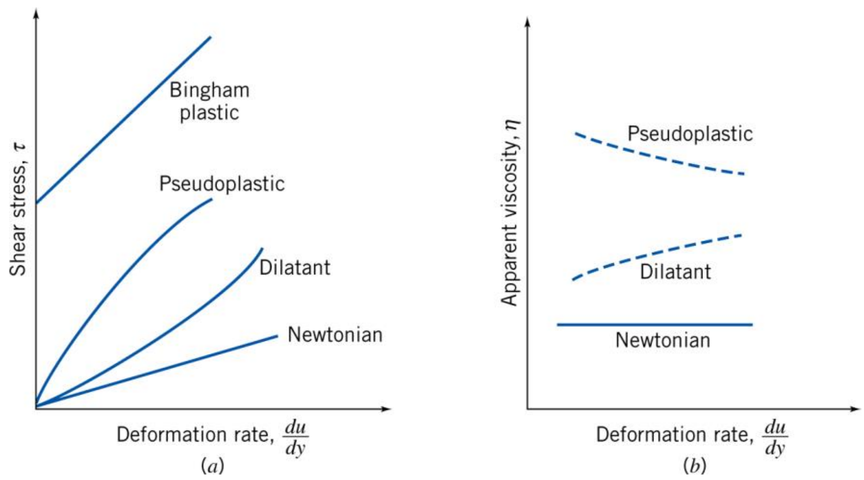
In there fluid shear stress-deformation rate (shear strain) relation may be represented by the power law model,

$$\tau_{yx} = k\left(\frac{du}{dy}\right)^n$$ n: flow behavior index, k: consistency index

If the above equation is written in the form

$$\tau_{yx} = k\left|\frac{du}{dy}\right|^{n-1}\frac{du}{dy} = \eta\frac{du}{dy}$$

then $\eta = k\left|\frac{du}{dy}\right|^{n-1}$ is referred to as the apparent viscosity.



Pseudoplastic Fluid (Shear thinning): apparent viscosity decreases with increasing deformation rate.

Example: polymer solutions, ketchup

Dilatant (Shear thickening): apparent viscosity increases with increasing deformation rate.
Example: sand suspension

Bingham plastic: deformation (flow) does not begin until a finite stress is applied.
Example: toothpaste, drilling muds, clay suspensions

Rheopectic fluid: apparent viscosity increases with time under constant shear stress.

Thixotropic fluid: apparent viscosity decreases with time under constant shear stress.
Example: paints

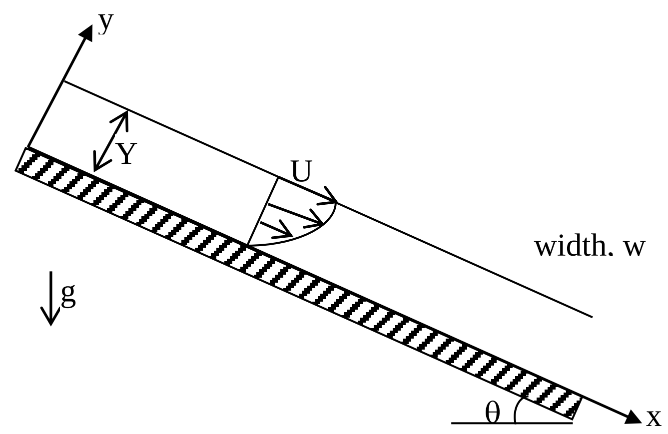
Viscoelastic fluid: fluid which partially returns to original shape when the applied stress is released.

Dependency of Viscosity on Temperature

In liquids, viscosity decreases with increasing temperature. This is a result of the fact that the distance between liquid molecules increases with increasing temperature, and hence cohesion between molecules decreases.

In gases, resistance to shear force depends on the momentum transfer between molecules with increasing temperature, motion of the gas molecules increases and hence momentum transfer increases, as a result viscosity increases.

Expamle: Consider a fluid flowing on an inclined surface. Its velocity profile is $u(y) = U\left[2\frac{y}{Y} - \left(\frac{y}{Y}\right)^2\right]$. Find shear stress at y=0, Y/2, Y.



For Newtonian fluid,

$$\tau_{yx} = \mu \frac{du}{dy} = \mu U \frac{d}{dy}\left[2\frac{y}{Y} - \left(\frac{y}{Y}\right)^2\right] = \mu \frac{U}{Y}\left[2 - 2\frac{y}{Y}\right]$$

Shear stresses at various locations y-locations are

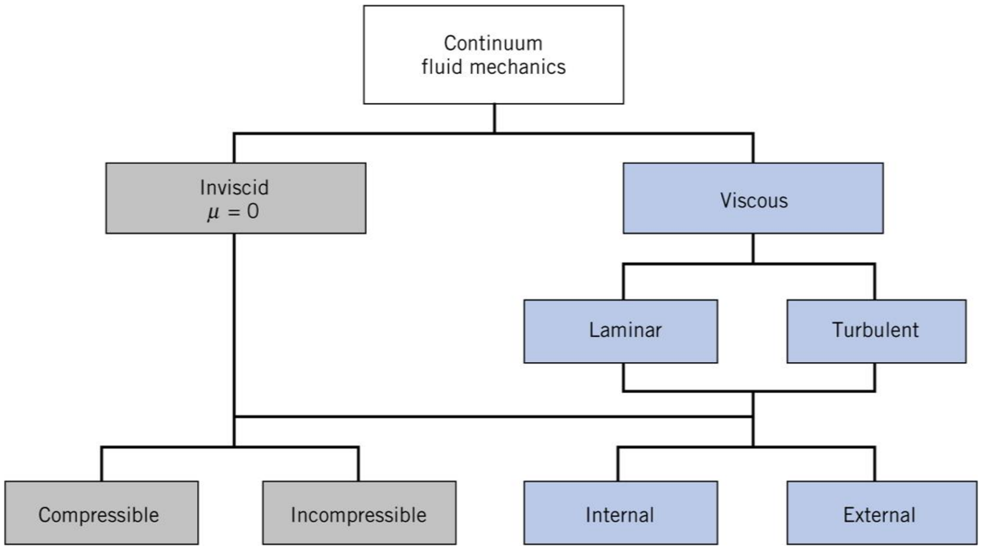
$$\tau_{yx}\Big|_{y=0} = 2\mu \frac{U}{Y}$$

$$\tau_{yx}\Big|_{y=Y/2} = \mu \frac{U}{Y}$$

$$\tau_{yx}\Big|_{y=Y} = 0 \quad \Leftarrow as\, the\, air\, above\, the\, liquid\, exerts\, a\, neglibile\;\, force\, on\, the\, liquid.$$

DESCRIPTION AND CLASSIFICATION OF FLUID MOTIONS

Since there is much overlap in the types of flow fields encountered, there is no universally accepted classification scheme. One possible classification,

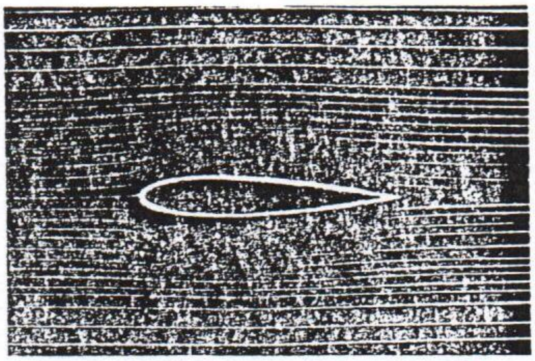


Viscous and Inviscid Flows

In an inviscid flow, the fluid viscosity, μ, ,s assumed to be zero. Fluids with zero viscosity do not exist; however, there are many problems where an assumption that μ=0 will simplify the analysis, and at the same time lead to meaningful results.

All fluids possess viscosity and consequently all flows are viscous.

Figure 3.4 Smoke marks streaklines in flow over an airfoil in a wind tunnel (from the "NCFMF Book of Film Notes," 1974, The MIT Press with Education Development Center, Inc., Newton, Mass.).



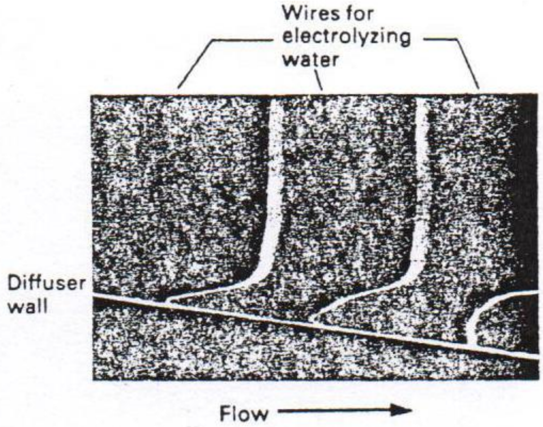


Figure 3.6 Timelines marked by hydrogen bubbles in water flow through a diffuser (from the "NCFMF Book of Film Notes," 1974, The MIT Press with Education Development Center, Inc., Newton, Mass.).

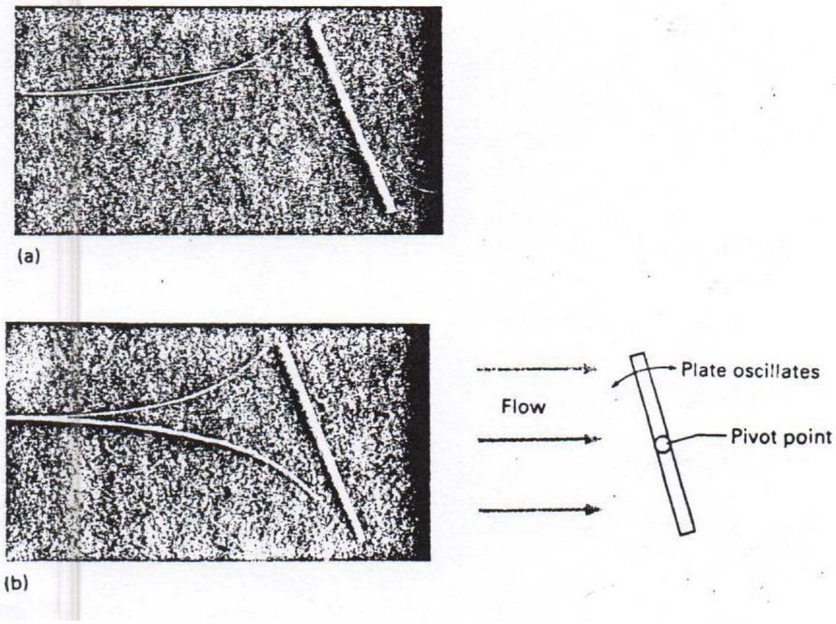
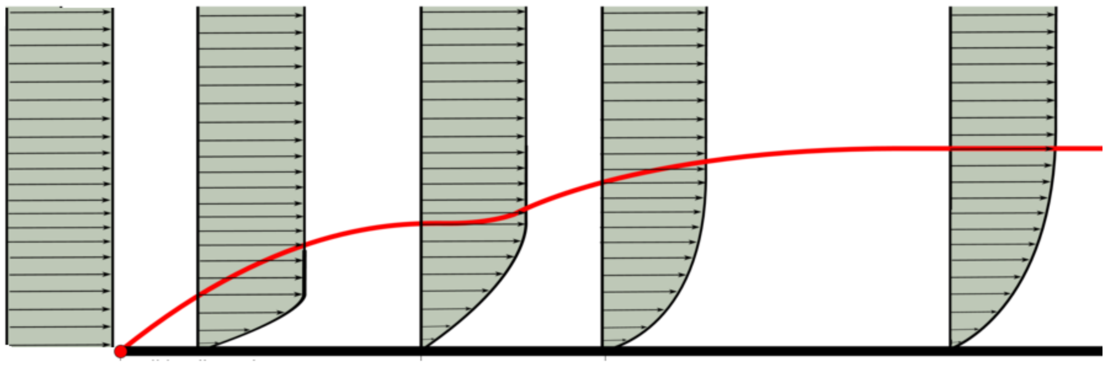


Figure 3.5 Difference between streamlines, streaklines, and pathlines in unsteady flow over an oscillating plate: (a) streamline (dotted) and pathline; (b) streamline and streakline. (From the "NCFMF Book of Film Notes," 1974, The MIT Press with Education Development Center, Inc., Newton, Mass.).



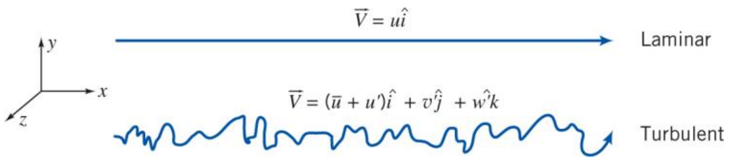
In any viscous flow, the flow in direct contact with a solid boundary has the same velocity as the boundary itself. There is no slip at the boundary.

Laminar and Turbulent Flows

The laminar flow is characterized by smooth motion of fluid particles in laminae or layers.

The turbulent flow is characterized by random, three-dimensional motions of fluid particles superimposed on the mean motion.

In laminar flow there is no macroscopic mixing of adjacent fluid layer.



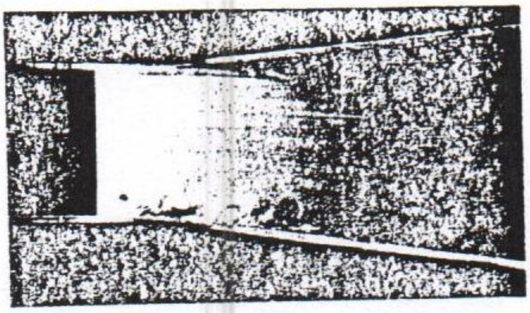


Figure 3.11 Water flowing in a diverging channel; flow is made visible by generating hydrogen bubbles from a wire spanning the channel (from the "NCFMF Book of Film Notes," 1974, The MIT Press with Education Development Center, Inc., Newton, Mass.).

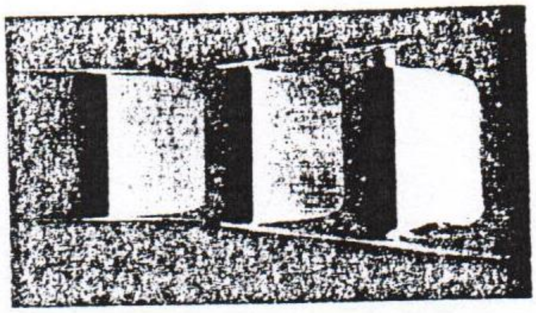
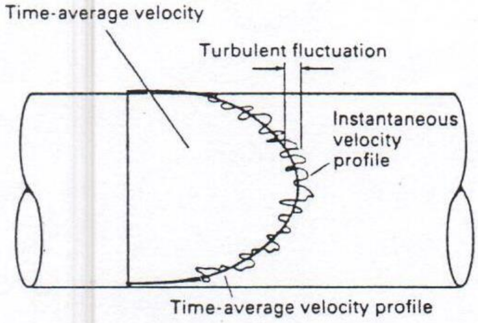


Figure 3.12 Velocity profiles at three locations in water flow in a diverging channel (from the "NCFMF Book of Film Notes," 1974, The MIT Press with Education Development Center, Inc., Newton, Mass.).

Figure 3.18 Instantaneous velocity profile in pipe flow; fluctuating velocity is superimposed on time-average velocity.



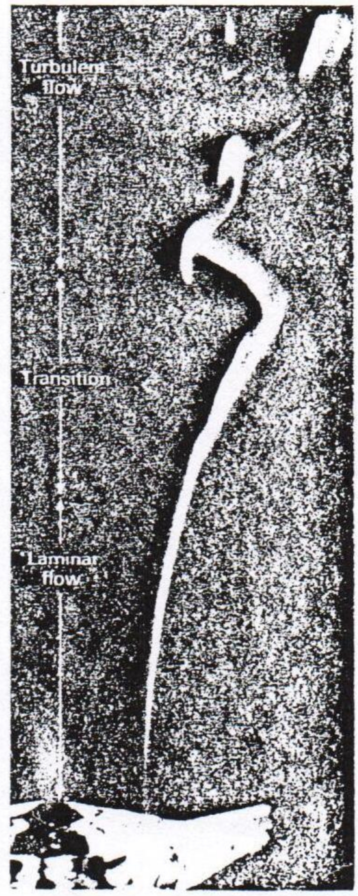


Figure 3.19 A plume of cigarette smoke illustrates laminar flow, transition, and turbulent flow (from the "NCFMF Book of Film Notes," 1974, The MIT Press with Education Development Center, Inc., Newton, Mass.).

FLUID STATICS

In this chapter, an expression for the pressure distribution in a stationary body of fluid will be derived, and the pressure forces acting on submerged surfaces will be studied.

In fluids at rest, there is no relative motion between fluid particles. Hence there is no shear stress acting on fluid elements. Fluids, which are at rest, are only able to sustain normal stresses. In fluids undergoing rigid-body motion, a fluid particle retains its identity and there is no relative motion between the particles. Hence, in fluids undergoing rigid-body motion only stress component present is the normal stress.

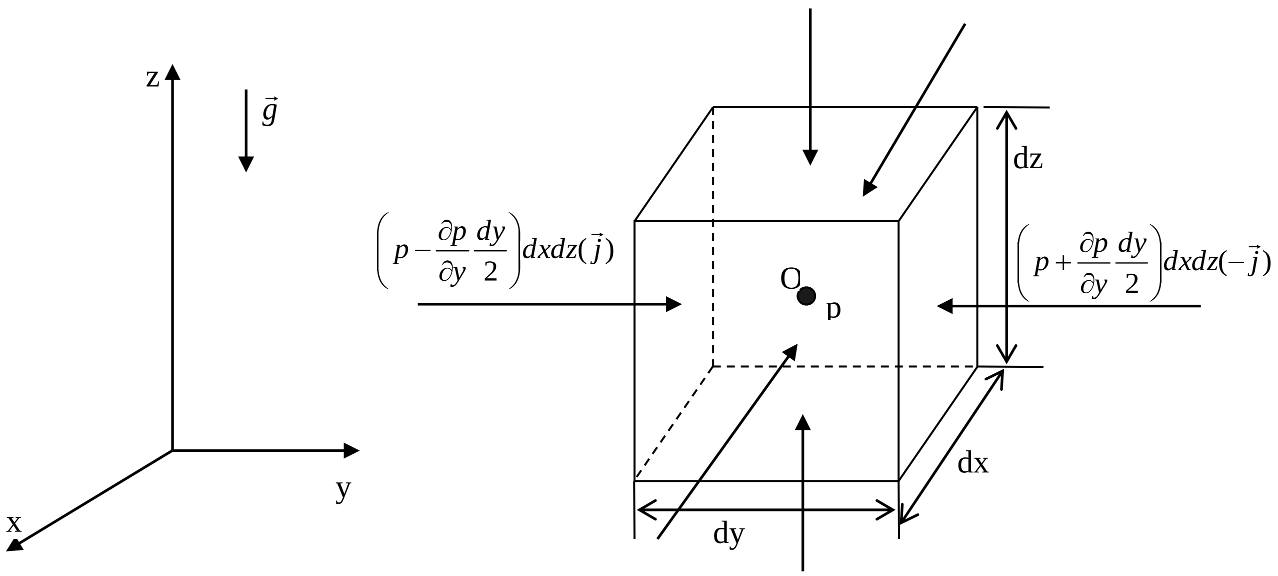
THE BASIC EQUATION OF FLUID STATICS

Our primary objective is to obtain an equation that will enable us to determine the pressure field within the fluid.

Consider a differential element of mass dm, with sides dx, dy, and dz. The fluid element is stationary relative to stationary coordinate system.

Two types of force may be acting on the fluid element.

- body force ← gravitational force
- surface force ← pressure force



In general P=P(x,y,z,t)

The force acting on fluid element shown,

$$d\vec{F} = d\vec{F}_B + d\vec{F}_s \qquad (1)$$

$$d\vec{F}_B = dm\vec{g} = d\forall\rho\vec{g} = dxdydz\vec{g} \qquad (2)$$

Let the pressure at the center O, of the element be P(x,y,z,t). To determine the pressure at each of the six forces of the element, we use Taylor series expansion about the point O. The pressure at the left face of the differential element is

$$p_L = p + \frac{\partial p}{\partial y}(y_L - y) = p + \frac{\partial p}{\partial y}\left(-\frac{dy}{2}\right) = p - \frac{\partial p}{\partial y}\frac{dy}{2}$$

Similarly on the right face

$$p_R = p + \frac{\partial p}{\partial y}(y_R - y) = p + \frac{\partial p}{\partial y}\frac{dy}{2}$$

Pressure forces on the other forces of the element are obtained in the same way. Combining all such forces gives the net surface force acting on the element

$$\begin{aligned} d\vec{F}_S &= \left(-\frac{\partial p}{\partial x}dx\right)dydz\vec{i} + \left(-\frac{\partial p}{\partial y}dy\right)dxdz\vec{j} + \left(-\frac{\partial p}{\partial z}dz\right)dxdy\vec{k} \\ &= \left(-\frac{\partial p}{\partial x}\vec{i} - \frac{\partial p}{\partial y}\vec{j} - \frac{\partial p}{\partial z}\vec{k}\right)dxdydz \\ &= -\left(\frac{\partial p}{\partial x}\vec{i} + \frac{\partial p}{\partial y}\vec{j} + \frac{\partial p}{\partial z}\vec{k}\right)dxdydz \end{aligned}$$

The term in parentheses is called the gradient of the pressure is simply pressure gradient and can be written grad P or ∇P. In rectangular coordinate system,

$$grad\,P \equiv \nabla P \equiv \left(\frac{\partial p}{\partial x}\vec{i} + \frac{\partial p}{\partial y}\vec{j} + \frac{\partial p}{\partial z}\vec{k}\right)$$

$$\therefore d\vec{F}_S = -grad\,P\,dxdydz = -\nabla P dxdydz \qquad (3)$$

$$grad\,P = \nabla P = -\frac{d\vec{F}_S}{dxdydz}$$

Physically the gradient of pressure is negative to the surface force per unit volume due to the pressure. We note that the level of pressure is not important in evaluating the net pressure force. Instead, what matters is the rate at which pressure changes occur with distance, the pressure gradient.

Combining equations (2) and (3) in Eq. (3)

$$\begin{aligned} d\vec{F} &= d\vec{F}_S + d\vec{F}_B \\ &= \left(-\,grad\,P + \rho\vec{g}\right)dxdydz \end{aligned}$$

Or on unit volume base

$$\frac{d\vec{F}}{d\forall} = \frac{d\vec{F}}{dxdydz} = -grad\,P + \rho\vec{g}$$

For a fluid particle, Newton's second law of motion gives $d\vec{F} = dm\vec{a} = \rho d\forall\vec{a}$. But for a static fluid, the acceleration $\vec{a}$ is zero. Thus,

$$d\vec{F} = \left(-\,grad\,P + \rho\vec{g}\right)dxdydz = 0$$

or

$$-\underbrace{grad\,P}_{\text{pressure force per unit volume at a point}} + \underbrace{\rho\vec{g}}_{\text{body force per unit volume at a point}} = 0$$

Components of this vector equation are

$$-\frac{\partial p}{\partial x} + \rho g_x = 0$$

$$-\frac{\partial p}{\partial y} + \rho g_y = 0$$

$$-\frac{\partial p}{\partial z} + \rho g_z = 0$$

Above equations describe the pressure variation in each of the three coordinate directions in a static fluid. To simplify further, it is logical to choose a coordinate system such that the gravity vector is aligned with one of the axes. If the coordinate system is chosen such that z-axis is directed vertically, then $g_x = 0, g_y = 0$ *and* $g_z = -g$

$$\left.\begin{aligned}\frac{\partial p}{\partial x}&=0\\ \frac{\partial p}{\partial y}&=0\\ \frac{\partial p}{\partial z}&=-\rho g\end{aligned}\right\}\Rightarrow \text{pressure is only function of z}$$

$$\therefore \frac{\partial p}{\partial z}\equiv\frac{dp}{dz}$$

$$\frac{dp}{dz}=-\rho g$$

or

$$\frac{dp}{dz}=-\gamma \qquad \text{(4) Basic equation of fluid statics}$$

Note: The pressure does not vary in a horizontal direction. The pressure increases if we go down and decreases if we go up.

PRESSURE VARIATION IN A CONSTANT-DENSITY FLUID

If the density of the fluid is constant, we can easily integrate Eq.(4) to give

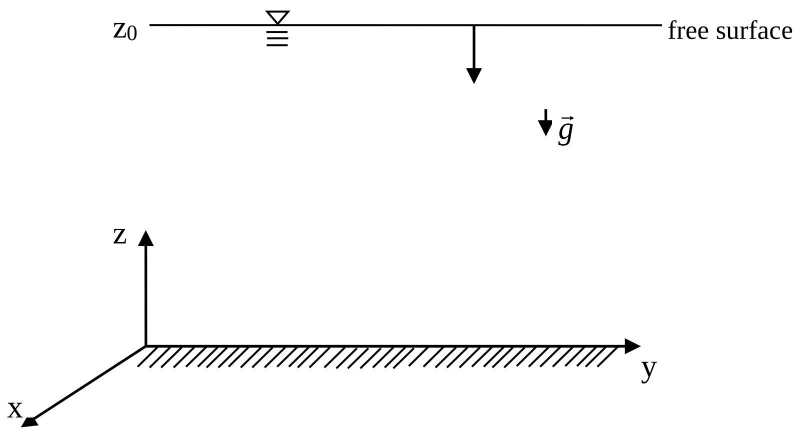
$$\frac{dp}{dz}=-\rho g$$

$$\int_{p_0}^{p}dp=-\int_{z_0}^{z}\rho g dz$$

$$p-p_0=-\rho g(z-z_0)$$

$$or$$

$$p-p_0=\rho g(z_0-z)$$

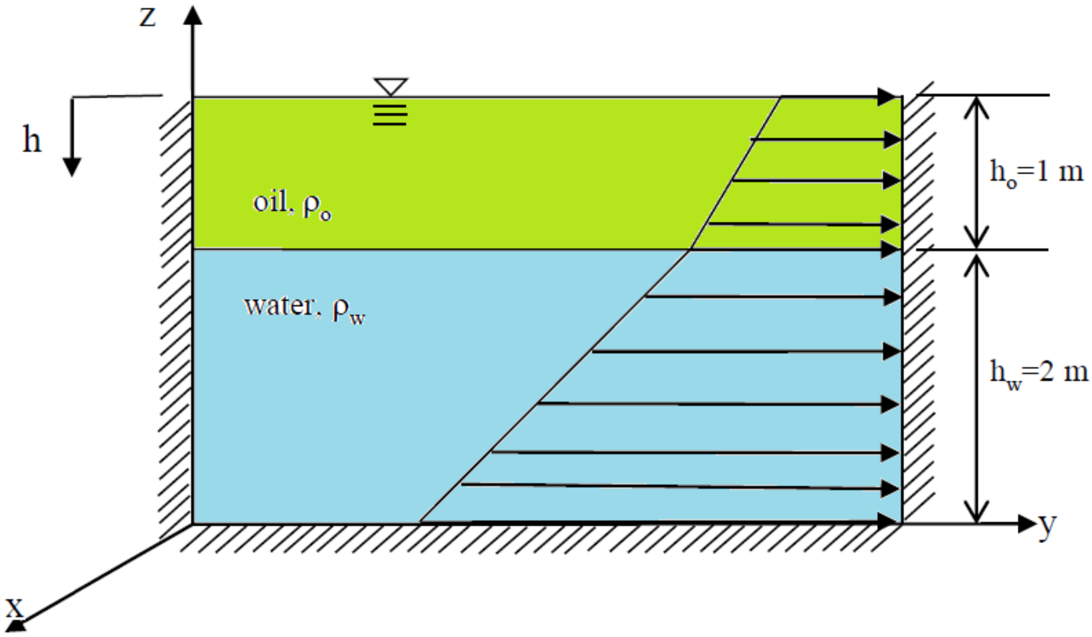


For liquids, it is often convenient to take the origin of the coordinate system at the free surface, and measure the distance as positive downward from the free surface with h measured positive downward, the

$$z_0-z=h$$

$p=p_0+\rho gh$ is called hydrostatic pressure where p_0 is the pressure at the free surface of the liquid.

Example: A tank which is exposed to the atmosphere, contains 2 m of water covered with 1 m of oil. The density of water and oil are 1000 kg/m^3 and 830 kg/m^3, respectively. Find the pressure at the interface and at the bottom of the tank. Also determine the pressure distribution at the tank. The atmospheric pressure is 101.325 kPa.



Basic equation of fluid statics, $\dfrac{dp}{dz} = -\rho g$

For ρ=constant, $p = p_0 + \rho g h$, pressure at any point in the fluid.

$$p_{\text{int}} = P_a + \rho_o g h_o = 101325\left[\frac{N}{m^2}\right] + 830\left[\frac{kg}{m^3}\right] 9.81\left[\frac{m}{s^2}\right] 1[m]$$

$$p_{\text{int}} = 109467.3\left[\frac{N}{m^2}\right]$$

$$p_b = p_{\text{int}} + \rho_w g h_w = 109467.3\left[\frac{N}{m^2}\right] + 1000\left[\frac{kg}{m^3}\right] 9.81\left[\frac{m}{s^2}\right] 2[m]$$

$$p_b = 129087.3\left[\frac{N}{m^2}\right]$$

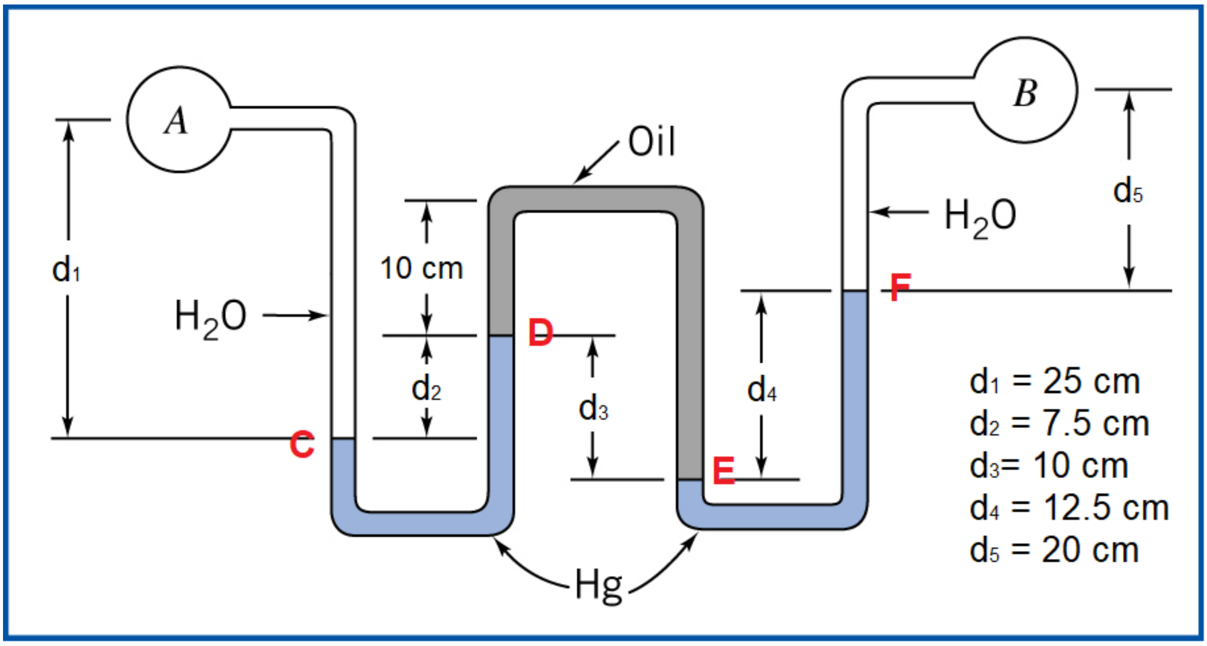
Variation of the pressure in oil is

$$p = p_{atm} + \rho_o gh = 101325\left[\frac{N}{m^2}\right] + 830\left[\frac{kg}{m^3}\right]9.81\left[\frac{m}{s^2}\right]h[m] = 101325 + 8142.3h \qquad for\ \ 0 \le h \le h_o$$

Variation of the pressure in water is

$$p = p_{\text{int}} + \rho_w gh = 109467.3\left[\frac{N}{m^2}\right] + 1000\left[\frac{kg}{m^3}\right]9.81\left[\frac{m}{s^2}\right]h[m] = 109467.3 + 9810h \qquad for\ \ 0 \le h \le h_w$$

Example: Water flows through pipes A and B. Oil, with specific gravity 0.8, is in the upper portion of the inverted U. Mercury (specific gravity 13.6) is in the bottom of the manometer bends. Determine the pressure difference, P_A-P_B.



Find: the pressure difference between A and B, p_A-p_B = ?

Given:

ρ_{H_2O} =1000 kg/m^3

$SG = \dfrac{\rho}{\rho_{H_20}} \quad \rightarrow \qquad \rho_{Hg}$ =13.6*1000= 13600 kg/m^3

ρ_{oil} =0.8*1000= 800 kg/m^3

Basic equation $\dfrac{dp}{dz} = -\dfrac{dp}{dh} = -\rho g \quad \Rightarrow \qquad dp = -\rho g dh$

$$\int_1^2 dp = \int_1^2 -\rho g dh$$

$$p_2 - p_1 = -\rho g(h_2 - h_1)$$

Beginning at point A and applying the above equation between successive points gives

$p_C - p_A = +\rho_{H2O} g d_1$ $\quad$ $p_E - p_D = +\rho_{oil} g d_3$ $\quad$ $P_B - P_F = -\rho_{H2O} g d_5$

$p_D - p_C = -\rho_{Hg} g d_2$ $\quad$ $p_F - p_E = -\rho_{Hg} g d_4$

$$\begin{aligned}
p_A - p_B &= (p_A - p_C) + (p_C - p_D) + (p_D - p_E) + (p_E - p_F) + (p_F - p_B) \\
&= -\rho_{H2O} g d_1 + \rho_{Hg} g d_2 - \rho_{oil} g d_3 + \rho_{Hg} g d_4 + \rho_{H2O} g d_5 \\
&= 9.8\left(-1000 \times 25 + 13600 \times 7.5 - 800 \times 20 + 13.6 \times 12.5 + 1000 \times 20\right) \times 10^{-2} \\
&= 25407.90\, Pa = 25.405\, kPa
\end{aligned}$$

Pressure Variation in a Varible-Density Fluid

If the density is variable, we must relate it relate to the pressure /or elevation before we can integrate the equation.

$$\frac{dp}{dz} = -\rho g$$

A common case might involve an ideal gas. In such gases, density can be expressed as a function of pressure and temperature. Pressure and density of liquids are related by the bulk compressibility modulus or modulus of elasticity.

$$E_v = \frac{dp}{d\rho / \rho} \quad \Rightarrow dP = E_v \frac{d\rho}{\rho}$$

If the bulk modulus is assumed to be a constant, then the density is only a function of the pressure.

$$\int_{P_0}^{P} dp = \int_{\rho_0}^{\rho} E_v \frac{d\rho}{\rho} \qquad \Rightarrow \qquad p - p_0 = E_v \ln \frac{\rho}{\rho_0} \qquad \Rightarrow \qquad p = p_0 + E_v \ln \frac{\rho}{\rho_0}$$

$$\left.\begin{aligned} \frac{dp}{dz} &= -\rho g \\ dp &= E_v \frac{d\rho}{\rho} \end{aligned}\right\} \Rightarrow \qquad \int_{\rho_0}^{\rho} \frac{d\rho}{\rho^2} = -\int_{z_0}^{z} \frac{g}{E_v} dz$$

$$\frac{1}{\rho_0} - \frac{1}{\rho} = -\frac{g}{E_v}\left(z - z_0\right)$$

$$\frac{1}{\rho_0} - \frac{1}{\rho} = +\frac{g}{E_v} h \qquad \Rightarrow \qquad \frac{1}{\rho} = \frac{1}{\rho_0} - \frac{g}{E_v} h$$

$$\frac{\rho_0}{\rho} = 1 - \frac{\rho_0 g}{E_v} h$$

$$\therefore \; p = p_0 + E_v \ln \frac{1}{1 - \frac{\rho_0 g}{E_v} h}$$

Example: The pressure, temperature and density of standard atmosphere at the sea level are 101.325 kPa, 15.2 °C, and 1.225 kg/m^3, respectively. Calculate the percent error introduced into the elevation of 8 km, by assuming the atmosphere.

a) to be incompressible
b) to be isothermal
c) to be isentropic
d) linearly decreasing temperature with a temperature decrease of -0.0065 K/m.

The actual pressure at an elevation of 8 km is known to be 35.656 kPa. The gas constant of air is 287 J/kgK.

a) Incompressible air, ρ=constant

$$\frac{dp}{dz} = -\rho g \;\Rightarrow\; p = p_0 - \rho_0 g z$$

p= 101325 N/m^2-(1.225 kg/m^3)(9.81 m/s^2)(8000)(m)

p= 5187 N/m^2

$$\%\, Error \;=\; \frac{35656\text{-}5187}{35656} \times 100 = 85.45\%$$

b) Isothermal

$$\rho = \frac{p}{RT}$$

$$\frac{dp}{dz} = -\rho g$$

$$\frac{dp}{P} = -\frac{g dz}{RT}$$

$$\ln \frac{p}{p_0} = -\frac{g}{RT_0}\left(z - z_0\right)$$

$$p = p_0 e^{-\frac{g}{RT_0}\left(z - z_0\right)}$$

$$p = 101325 \exp\left(-\frac{9.81(8000-0)}{287(288.2)} \right) = 39232.86\ N/m^2$$

$$\%\ Error\ = \ \frac{39232 - 35656}{35656} \times 100 = 10.03\%$$

c) Isentropic

$$p_1 V_1^k = p_2 V_2^k$$

$$p_1 \frac{1}{\rho_1^k} = p_2 \frac{1}{\rho_2^k} \Rightarrow \frac{p}{\rho^k} = \frac{p_0}{\rho_0^k} \Rightarrow \rho = \left(\frac{p}{p_0}\right)^{1/k} \rho_0$$

$$\frac{dp}{dz} = -\rho g$$

$$\frac{dp}{dz} = -\left(\frac{p}{P_0}\right)^{1/k} \rho_0 g$$

$$\int_{P_0}^{P} \frac{dp}{p^{1/k}} = \int_{z}^{z_0} - P_0^{1/k} \rho_0 g$$

or

$$p = p_0 \left[1 - \frac{(k-1)\rho_0 g}{kP_0} \left(z - z_0\right) \right]^{k/k-1}$$

$$P = 101325 \left[1 - \frac{(1.4-1)(1.225)(9.81)}{1.4(101325)} \left(8000 - 0\right) \right]^{1.4/1.4-1} = 33503.66 N/m^2$$

Percent error

$$\%\ Error\ = \ \frac{35656 - 33503.66}{35656} \times 100 = 6.04\%$$

d) Temperature decreases with increasing height

$$T = T_0 + mz, \qquad m < 0, \qquad m = -0.0065\,K/m$$

$$\rho = \frac{P}{RT} = \frac{P}{R(T_0 + mz)} \qquad \Rightarrow$$

$$\frac{dp}{dz} = -\rho g$$

$$\frac{dp}{dz} = -\frac{pg}{R(T_0 + mz)}$$

$$\int_{P_0}^{P} \frac{dp}{P} = -\frac{g}{R}\int_{z_0}^{z} \frac{dz}{(T_0 + mz)}$$

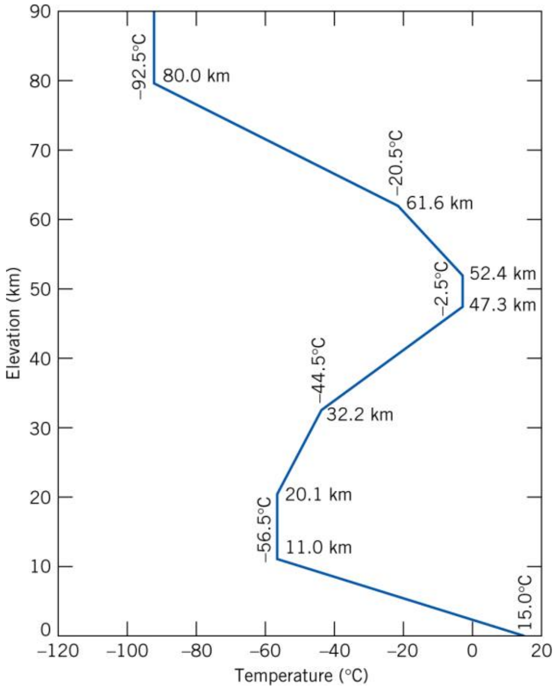
$$\ln\frac{p}{p_0} = -\frac{g}{Rm}\ln\frac{T_0 + mz}{T_0 + mz_0} \qquad \Rightarrow\ p = p_0\left(\frac{T_0 + mz}{T_0 + mz_0}\right)^{-\frac{g}{Rm}}$$

$$p = 101325\left(\frac{282.5 + (-0.0065)(8000)}{282.5 + (-0.0065)(0)}\right)^{-\frac{9.81}{287(-0.0065)}},$$

$$p = 35587.36\,N/m^2$$

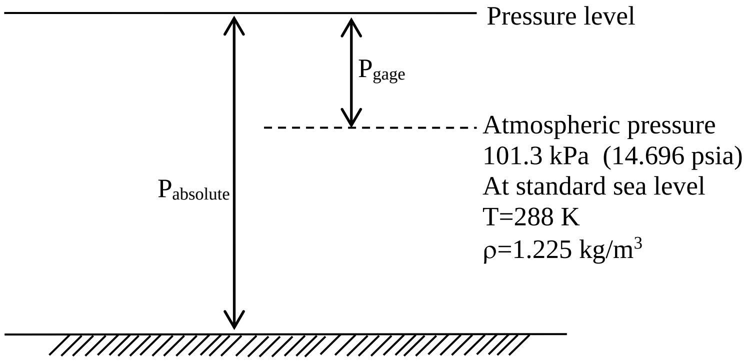
Percent error

$$\%\ Error \ = \ \frac{35656 - 35587}{35656} \times 100 = 0.19\%$$

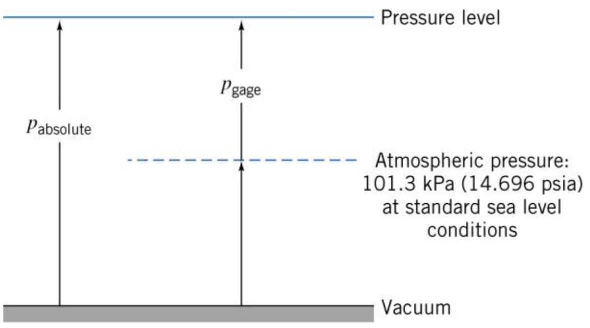


ABSOLUTE AND GAGE PRESSURE

Pressure values must be stated with respect to a reference level. If the reference level is a vacuum, pressures are termed as absolute. Pressure levels measured with respect to atmospheric pressure are termed gage pressure.



$p_{absolute} = p_{gage} + p_{atmosphere}$

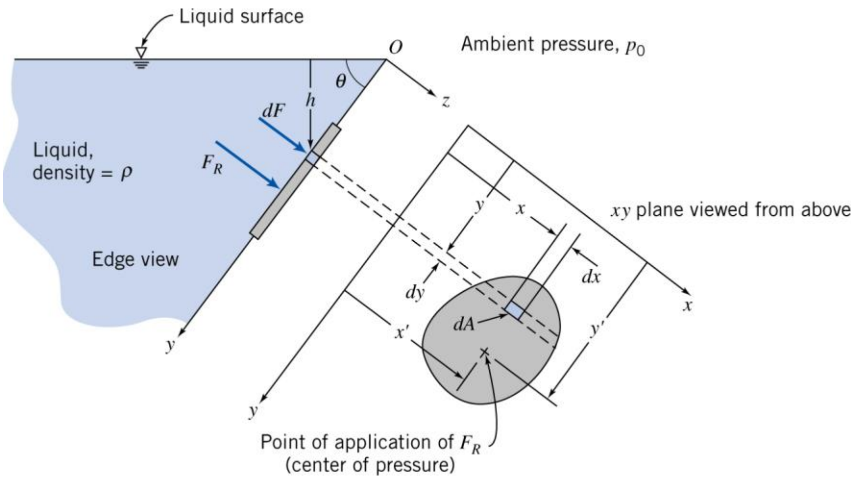


<u>**HYDROSTATIC FORCE ON SUBMERGED SURFACES**</u>

When a surface is in contact with a fluid, fluid pressure exerts a force on the surface. This force is distributed over the surface; however, it's often helpful in engineering calculations to replace the distributed force by a single resultant. To completely specify the resultant force we must determine its magnitude, direction and point of application.

We shall consider both plane and curved submerged surfaces.

1. HYDROSTATIC FORCE ON A PLANE SUBMERGED SURFACE



- Magnitude of resultant force

  $\left|\vec{F}_R\right| = ?$

- Point of application

  $x' = ?,\ y' = ?$

Force acting on surface $d\vec{A}$

$d\vec{F} = -pd\vec{A}$ - sign indicates that force $d\vec{F}$ acts againstthe surface $d\vec{A}$

The resultant force acting on the surface is found by summing the contribution of the infinitesimal forces over the entire area.

Thus,

$$\vec{F}_R = -\int_A p d\vec{A} \quad \text{..(1)}$$

In order to calculate the integral, both pressure, p, and the element area of $d\vec{A}$ must be expressed in terms of the same variables. The basic pressure-height relation for a static fluid can be written as

$$\frac{dp}{dh} = \rho g \,, \qquad \text{h is measured positive downward from the liquid free surface.}$$

$$\int_{p_0}^{p} dp = \int_{h=0}^{h} \rho g dh \quad \Rightarrow \quad p = p_0 + \rho g h \quad \text{..(2)}$$

p_0 is the pressure at liquid free surface (h=0)

This expression can be substituted into Eq. 1. Then to perform integration, h and DA should be expressed in terms of x and/or y. ($h = y \sin\theta$, θ = constant). Integration of Eq. 1 gives the resultant force due to the distributed pressure force.

The point of application of the resultant force must be such that the moment of the resultant force about any axis is equal to the moment of the distributed force about the same axis.

Let $\vec{r}'$ be the position vector of the point of application of the resultant force $\vec{F}_R$ and $\vec{r}$ be the position vector of any point on surface A.

$$\vec{r}' \times \vec{F}_R = \int \vec{r} \times d\vec{F}$$

$$\vec{r}' \times \vec{F}_R = -\int_A \vec{r} \times P d\vec{A}$$

According to coordinate system used,

....

$$\vec{r}' = x'\vec{\imath} + y'\vec{\jmath}$$

$$\vec{r} = x\vec{\imath} + y\vec{\jmath}$$

$$\vec{F}_R = -F_R \vec{k}$$

$$d\vec{A} = \vec{k} dA$$

$\therefore$

$$(x'\vec{\imath} + y'\vec{\jmath}) \times (-F_R \vec{k}) = -\int_A (x\vec{\imath} + y\vec{\jmath}) \times P dA\vec{k}$$

Evaluating the cross product, we obtain,

$$x'F_R\vec{j} + y'F_R\vec{\imath} = \int_A (xP\vec{j} + yP\vec{\imath})dA$$

Considering the components of this vector equation, we obtain

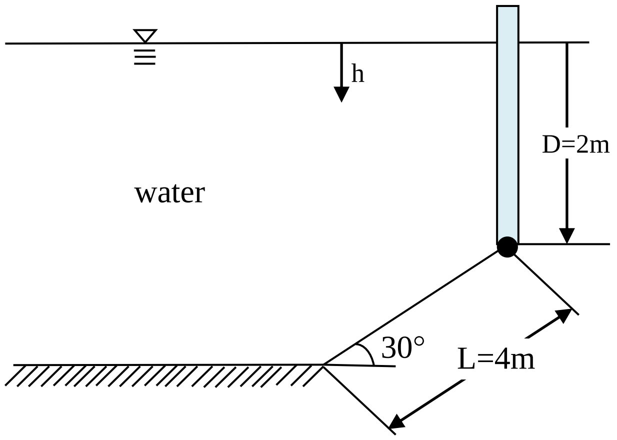
$$y'F_R = \int_A ypdA \Rightarrow y' = \frac{1}{F_R}\int_A ypdA$$

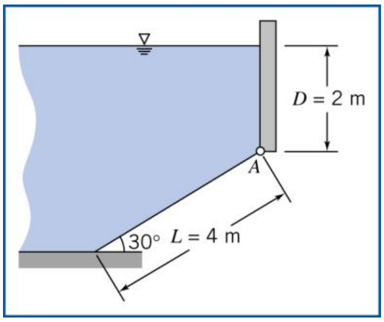
$$x'F_R = \int_A xpdA \Rightarrow x' = \frac{1}{F_R}\int_A xpdA$$

$$NOTE: F_R = \left|\vec{F}_R\right| = \int_A pdA \qquad \leftarrow \text{Magnitude of } \vec{\mathrm{F}}_\mathrm{R}$$

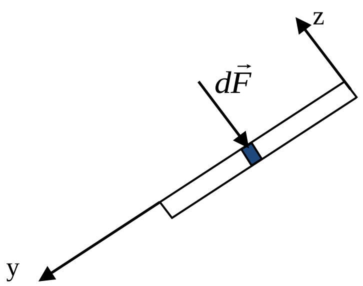
$-$ Direction of $\vec{\mathrm{F}}_\mathrm{R}$ is normal to the surface

Example: The inclined surface shown, hinged along A, is 5m wide. Determine the resultant force $\vec{F}_R$ of the water on the inclined surface.





w = 5m.



$$\vec{F}_R = -\int P d\vec{A}$$

$$d\vec{A} = w dy \vec{k}$$

We now need P as a function of y to perform the integration.

$$\left.\begin{aligned} p &= p_0 + \rho g h \\ p_0 &= p_{atm} \\ h &= D + y\sin 30^\circ \end{aligned}\right\} \Rightarrow p = p_a + (D + y\sin 30^\circ) g\rho$$

Since we are interested in the force of the water on the gate, then we drop P_a and obtain,

$$P = (D + y\sin 30^\circ) g\rho$$

$$\therefore \vec{F}_R = -\int p d\vec{A} = -\int_0^L \rho g (D + y\sin 30^\circ) w dy \vec{k}$$

$$= -\rho g w \left[Dy + \frac{y^2}{2}\sin 30^\circ \right]_0^L \vec{k} = -\rho g w \left[DL + \frac{L^2}{2}\sin 30^\circ \right]\vec{k}$$

$$= -999\left[\frac{kg}{m^3}\right] * 9.81\left[\frac{m}{s^2}\right] * 5[m] * \left[2[m] * 4[m] + \frac{16[m^2]}{2}\frac{1}{2}\right]\vec{k}$$

$$= -588\vec{k}\ kN \qquad \leftarrow \text{The force is in negative z - direction.}$$

Point of application of resultant force

$$F_R y' = \int_A ypdA$$

$$y' = \frac{1}{F_R}\int_A ypdA = \frac{1}{F_R}\int_0^L ypwdy = \frac{\rho g w}{F_R}\int_0^L y(D + y\sin 30^\circ)dy$$

$$y' = \frac{\rho g w}{F_R}\left[\frac{D}{2}y^2 + \frac{y^3}{3}\sin 30^\circ\right]_0^L = \frac{\rho g w}{F_R}\left[\frac{D}{2}L^2 + \frac{L^3}{3}\sin 30^\circ\right]$$

$$y' = 999\left[\frac{kg}{m^3}\right]*9.81\left[\frac{m}{s^2}\right]*\frac{5[m]}{588*10^3[N]}*\left[\frac{2[m]}{2}16[m^2] + \frac{64[m^3]}{3}\frac{1}{2}\right]$$

$$y' = 2.22[m]$$

$$x' = \frac{1}{F_R}\int_A xpdA$$

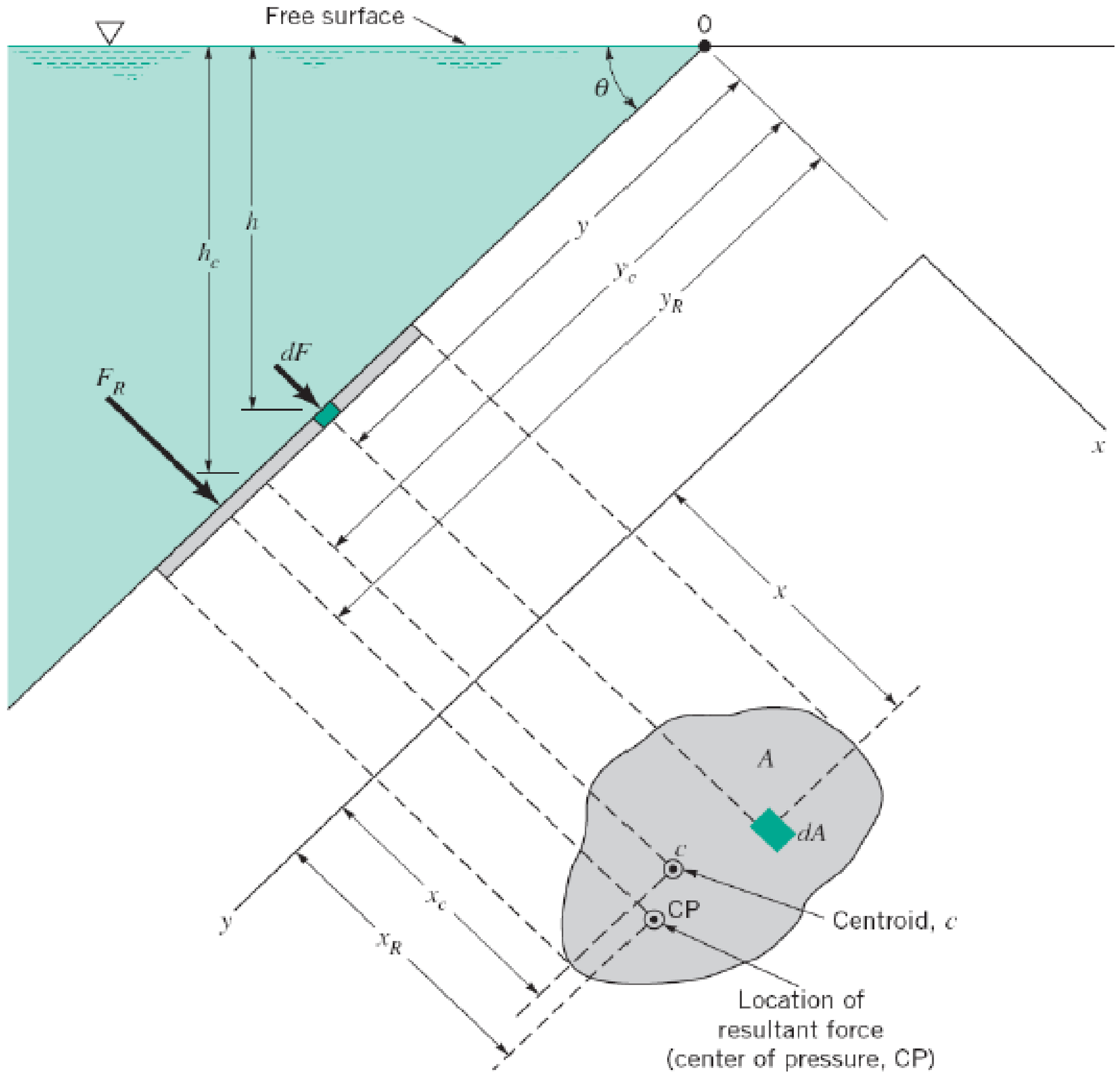
Since the area element is of constant width, *x=w/2*

$$x' = \frac{1}{F_R}\int_A \frac{w}{2}pdA = \frac{w}{2}\frac{1}{F_R}\int_A pdA = \frac{w}{2} = 2.5\,m.$$

Thus,

$r' = 2.5\vec{i} + 2.22\,\vec{j}\,[m]$. $\leftarrow$ line of action of $\vec{F}_R$

ALTERNATIVE APPROACH FOR CALCULATION OF HYDROSTIC FORCE (ALGEBRAIC EQUATIONS)



Note: Origin of the coordinate system is placed at the intersection of the plane of the gate and the free surface.

Now we will formulate an approach to determine the resultant hydrostatic force and its point of application. Consider the expressions developed before, i. e.

$$\vec{F}_R = -\int_A p d\vec{A}$$

Considering the free surface is open to atmosphere, the magnitude of the resultant force can be written as

$$\vec{F}_R = \int_A \rho g h dA = \int_A \rho g y \sin\theta dA = \rho g \sin\theta \int_A y dA = \rho g \sin\theta y_c A = \rho g h_c A$$

NOTE: $\int_A y dA = y_c A$ is the first moment of the area with respect to the x axis.

Where $\mathbf{y_c}$ is the y coordinate of the centroid of the area **A** measure from the x axis, which passes through **O**, and $\mathbf{y_c \sin\theta = h_c}$.

h_c is the vertical distance from the fluid surface to the centroid of the area.

Point of Application of the Resultant Force

Expressions for the coordinates of the point of application of the resultant force can be obtained by equating the moment of the resultant force to the moment of the distributed pressure force.

$$F_R y_R = \int_A y dF = \int_A \rho g \sin\theta y^2 dA$$

$$y_R = \frac{1}{F_R} \int_A \rho g \sin\theta y^2 dA = \frac{1}{\rho g \sin\theta y_c A} \int_A \rho g \sin\theta y^2 dA = \frac{\int_A y^2 dA}{y_c A}$$

$\int_A y^2 dA = I_x$ is **the *second moment of the area (moment of inertia)***, with respect to an axis formed by the intersection of the plane containing the surface and the free surface (*x* axis). Thus, we can write

$$y_R = \frac{I_x}{y_c A}$$

Using parallel axis theorem

$$I_x = I_{xc} + A y_c^2$$

where ***Ixc*** is the second moment of the area with respect to an axis passing through its ***centroid*** and parallel to the *x* axis. Thus,

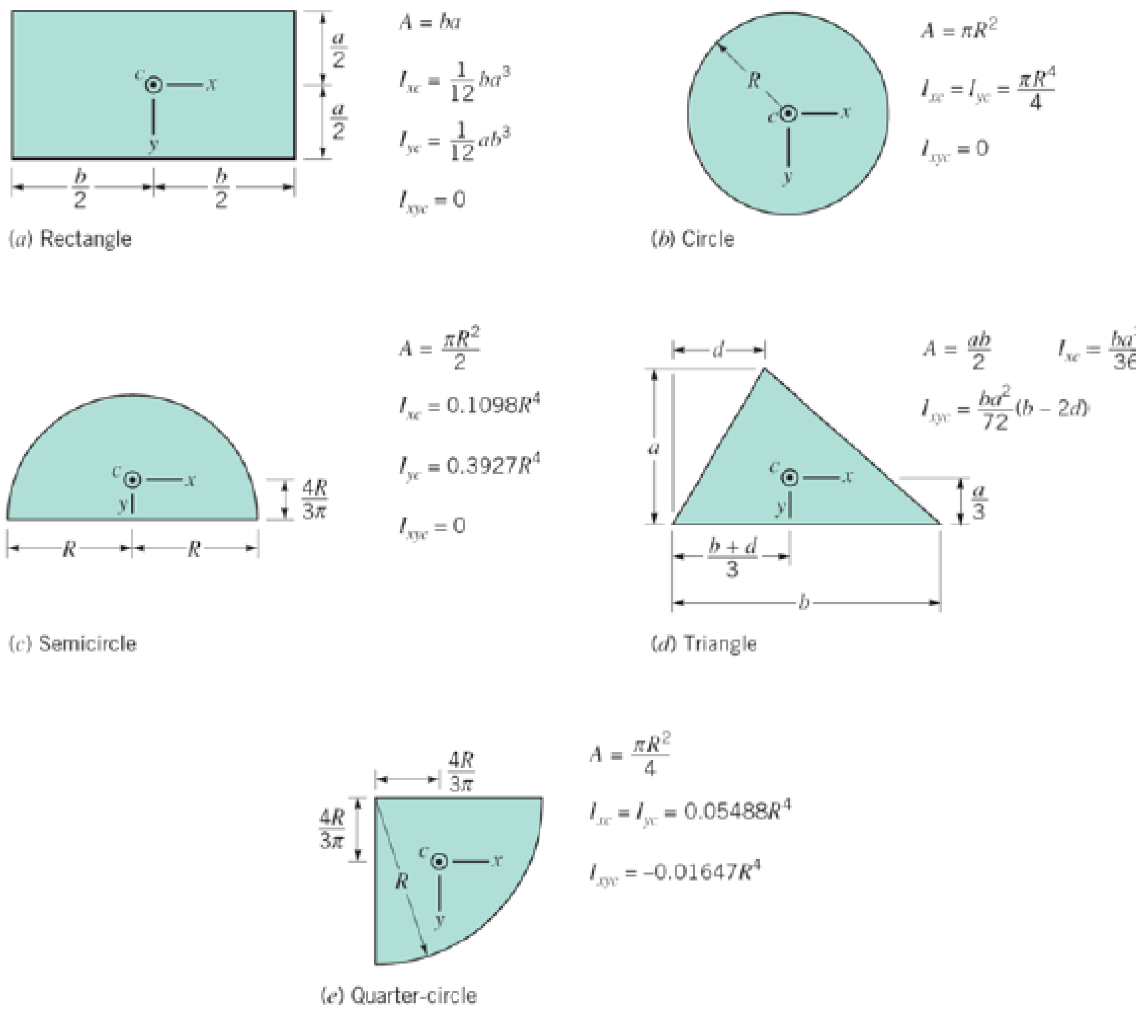
$$y_R = \frac{I_{xc}}{y_c A} + y_c$$

The ***x*** coordinate, x_R, for the resultant force can be determined in a similar manner as follows

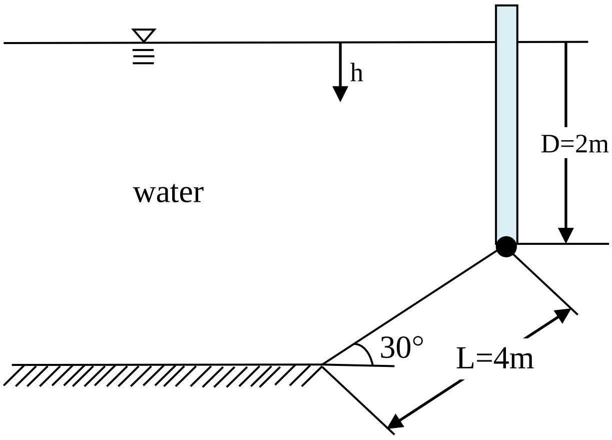
$$x_R = \frac{I_{xyc}}{y_c A} + x_c$$

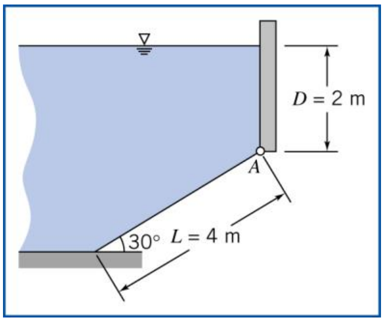
where ***Ixyc*** is the product of inertia with respect to an orthogonal coordinate system passing through the ***centroid* of the area**. The point through which the resultant force acts is called the ***center of pressure.***

Geometric properties of some common shapes

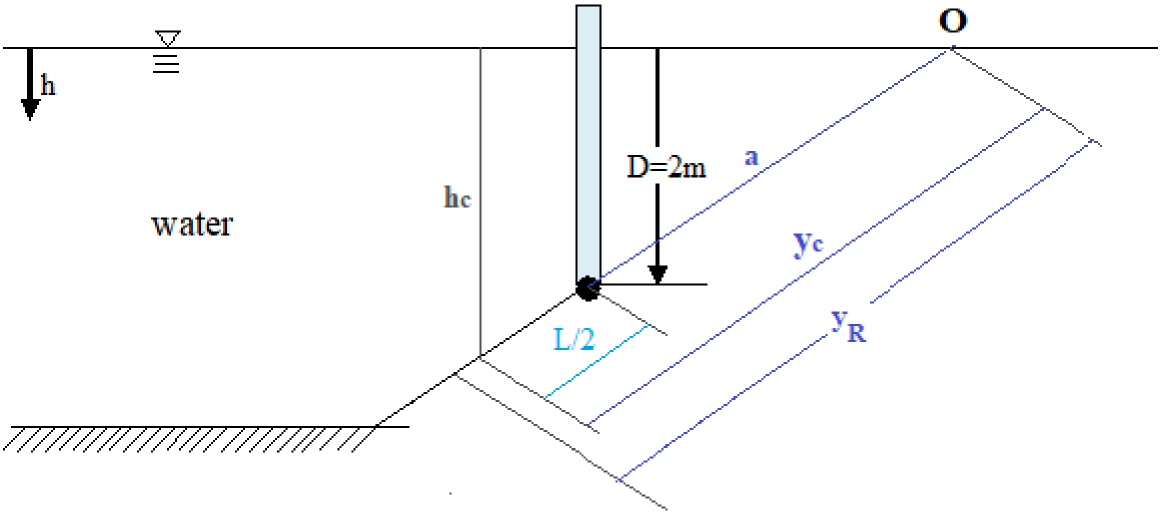


Example: Solve the previous example using the **algebraic equations** method.





w = 5m



$$a = \frac{D}{\sin 30} = 2D$$

$$h_c = D + \frac{L}{2}\sin 30$$

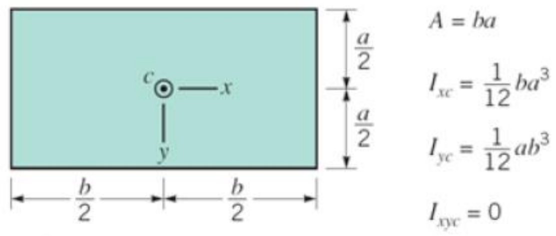
$$A = Lw$$

$$F_R = \rho g h_c A$$

$$F_R = \rho g\left(D + \frac{L}{2}\sin 30\right)(Lw) = 999\left[\frac{kg}{m^3}\right]9.81\left[\frac{m}{s^2}\right]\left(2 + \frac{4}{2}\sin 30\right)[m](4\cdot 5)\left[m^2\right]$$

$$F_R = 588011.4[N] = 588[kN]$$

$$y_R = \frac{I_{xc}}{y_c A} + y_c$$



(a) Rectangle

$$I_{xc} = \frac{1}{12}ba^3 = \frac{1}{12}wL^3$$

$$y_c = \frac{L}{2} + 2D$$

$$y_R = \frac{I_{xc}}{y_c A} + y_c = \frac{\frac{1}{12}wL^3}{\left(\frac{L}{2}+2D\right)(Lw)} + \left(\frac{L}{2}+2D\right)$$

$$y_R = \frac{1}{12}\frac{L^2}{\left(\frac{L}{2}+2D\right)} + \left(\frac{L}{2}+2D\right) = \frac{1}{12}\frac{4^2}{\left(\frac{4}{2}+2(2)\right)} + \left(\frac{4}{2}+2(2)\right)$$

$$y_R = 6.222[m]$$

$$x_R = \frac{I_{xyc}}{y_c A} + x_c$$

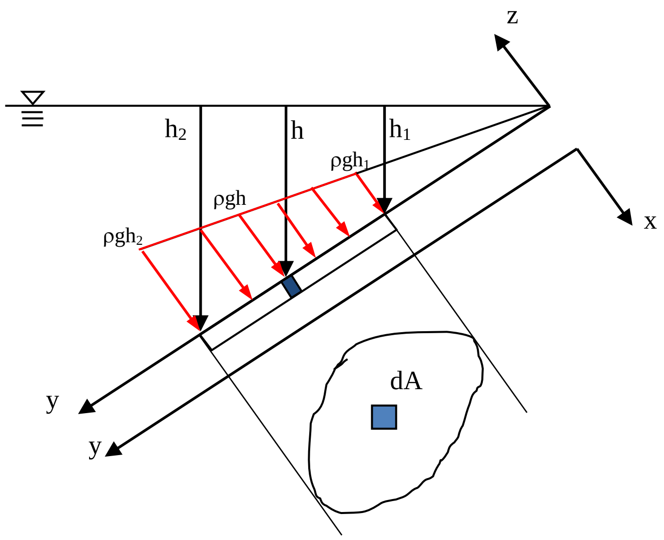
$$I_{xyc} = 0$$

$x_c = 2.5[m]$ depending on the coordinate system axis

$$x_R = 2.5[m]$$

PRESSURE PRISM METHOD

The concept of the pressure prism provides another tool for determining the magnitude and point of application of the resultant force on a submerged plane surface.



Considering the gage pressure at the free surface is zero, the infinitesimal pressure force, $d\vec{F}$, acting on the submerged plane surface is,

$$d\vec{F} = -PdA\vec{k} = -\rho ghdA\vec{k} = -d\forall_P \vec{k}$$

where dA and ρgh are infinitesimal base arae and imaginary height of the pressure prism, respectively. Thus, product of dA and ρgh represents the infinitesimal volume $d\forall_P$ of the pressure prism. After integration, the magnitude of the resultant force may be obtained as,

$$\vec{F}_R = \vec{k}\int_{\forall_P} d\forall_P = -\forall_P \vec{k}$$ $\forall_P$ is the volume of the prism.

Therefore, the magnitude of the resultant force acting on a submerged plane surface is equal to the volume of the pressure prism.

Point of application of the resultant force,

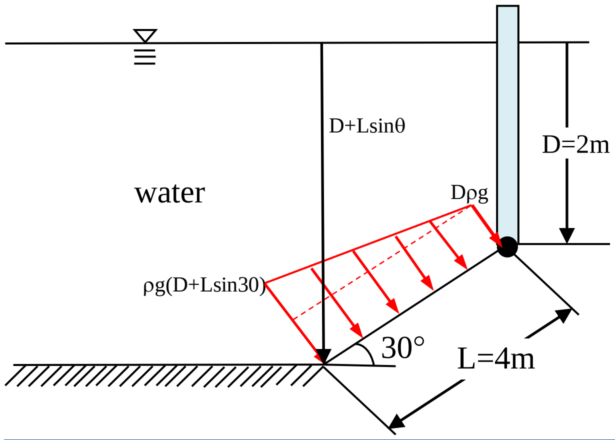
$$x' = \frac{1}{F_R}\int_A xPdA = \frac{1}{\forall_P}\int_A x\rho g h dA = \frac{1}{\forall_P}\int_{\forall_P} x d\forall = X_G$$

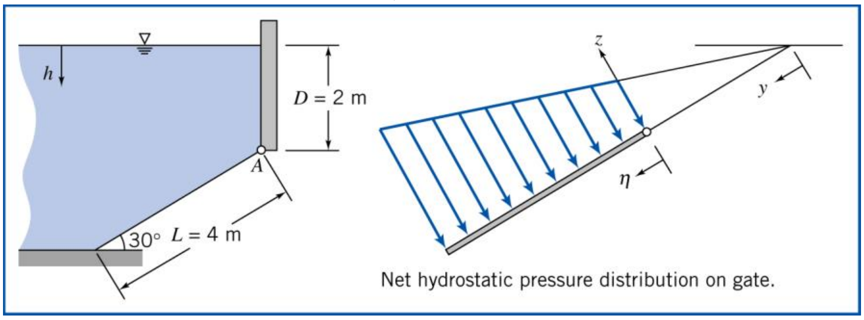
and

$$y' = \frac{1}{F_R}\int_A yPdA = \frac{1}{\forall_P}\int_A y\rho g h dA = \frac{1}{\forall_P}\int_{\forall_P} y d\forall = Y_G$$

where X_G and Y_G are the coordinates of the centroid of the pressure prism.

<u>Example</u>: solve the previous example using the pressure prism method.





$$\vec{F}_R = -\forall_P \vec{k}$$

$$\vec{F}_R = -\left[wL\frac{\rho g(D + L\sin 30) + \rho g D}{2}\right]\vec{k}$$

$$\vec{F}_R = -\rho g w L\left[D + \frac{L\sin 30}{2}\right]\vec{k}$$

$$\vec{F}_R = -588\,kN$$

Point of application of the resultant force

-The gate is symmetrical about its centroid axis.

$$\therefore\ x' = X_G = \frac{w}{2} = 2.5m$$

The y-coordinate of the point of application of the resultant force can be found considering the triangular pressure prisms. Let y_1' be the centroid of the rectangular pressure prism and y_2' be the centroid of the triangular pressure prism.

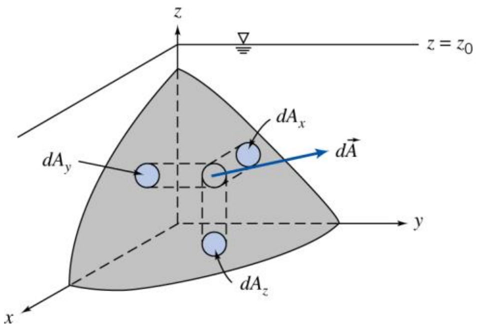
$$\therefore \qquad y' = Y_G$$

$$y'F_R = y_1'F_{R_1} + y_2'F_{R_2}$$

$$y'\forall_P = \left(\frac{L}{2}\right)(\rho g D w L) + \left(\frac{2}{3}L\right)\frac{\rho g L^2 w \sin 30}{2}$$

$$y' = 2.22\,m$$

HYDROSTATIC FORCE ON CURVED SUBMERGED SURFACES



Consider the infinitesimal curved surface shown in figure. The hydrostatic force on an infinitesimal element of a curved surface, $d\vec{A}$, acts normal to the surface. However, the differential pressure force on each element of the surface acts in a different direction because of the surface curvature.

Usually, to sum a series of force vetors acting in different directions, we sum the components of the vectors relative to a convenient system.

The pressure force acting on area element $d\vec{A}$ is

$$d\vec{F} = -p d\vec{A}$$

The resultant force is

$$\vec{F}_R = -\int_A p d\vec{A}$$

$\vec{F}_R$ can be written as

$$\vec{F}_R = \vec{i} F_{R_x} + \vec{j} F_{R_y} + \vec{k} F_{R_z}$$

where F_{R_x}, F_{R_y} and F_{R_z} are components of $\vec{F}_R$ in x,y and z directions.

To evaluate the component of the force in a given direction, we take the dot product of the force with the unit vector in the given direction. For example, taking the dot product of each side of the above equation with unit vector $\vec{i}$ gives

$$\vec{F}_R \cdot \vec{i} = -\int p d\vec{A} \cdot \vec{i}$$

$$F_{R_x} = -\int p dA_x$$

In general, magnitude of the component of the resultant force in the l direction is given by

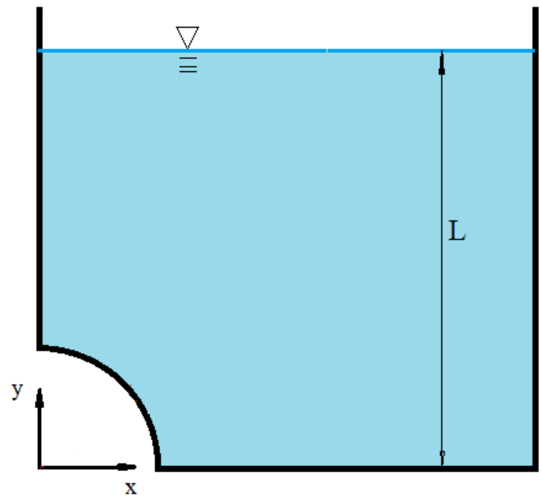
$$F_{R_l} = \int_{A_l} p dA_l$$

where dA_l is the projection of the area element dA on a plane perpendicular to l-direction.

The line of action of each component of the resultant force is found by recognizing that the moment of the resultant force component about a given axis must be equal to the moment of the corresponding distributed force component about the same axis.

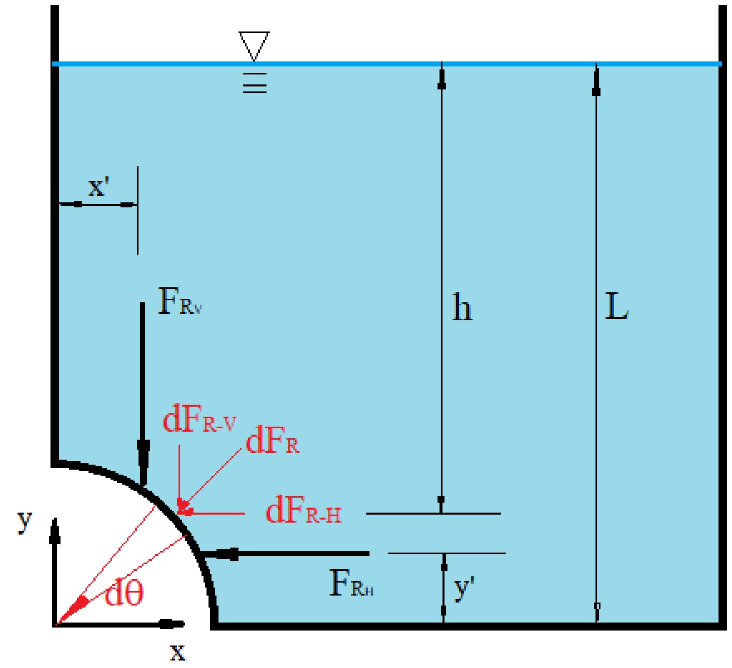
Because we are dealing with a curved surface, the lines of action of the components of the resultant force will not necessarily coincide; the complete resultant may not be expressed as a single force.

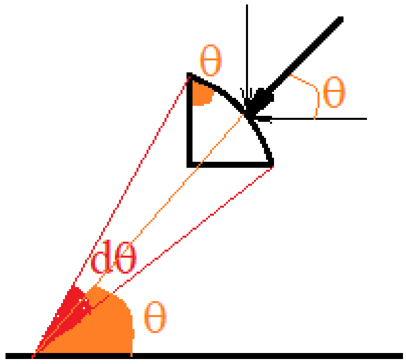
Example: An open tank which is shown in the figure is filled with an incompressible fluid of density, ρ. Determine the magnitudes and lines of action of the vertical and horizontal components of the resultant pressure force on the curved part of the tank bottom.



NOTE: the width of the tank is w.

FIND: $F_{R_H} = ?,\ F_{R_V} = ?$





Consider the area element $d\vec{A}$. The resultant force acting on this area element is

$$\vec{F}_R = -\int_A p\,d\vec{A}, \qquad dA = d\theta R w, \qquad p = p_0 + \rho g h$$

$$p_0 = p_{atm}$$

$$p = (L - R\sin\theta)\rho g$$

Horizontal component of this force is

$$dF_{R_H} = dF_R \cos\theta$$

Vertical component is

$$dF_{R_V} = dF_R \cos\left(\frac{\pi}{2} - \theta\right)$$

$$\therefore F_{R_H} = -\rho g w R \int_0^{\pi/2} (L - R\sin\theta)\cos\theta d\theta = -\rho g w R \int_0^{\pi/2} \left(L\cos\theta - \frac{R\sin 2\theta}{2}\right) d\theta$$

$$= -\rho g w R \left[L\sin\theta - \frac{R\cos 2\theta}{4}\right]_0^{\pi/2} = -\rho g w R \left[L - \frac{R}{4} - \frac{R}{4}\right] = -\rho g w R \left[L - \frac{R}{2}\right]$$

Similarly, vertical component of the resultant force

$$F_{R_v} = -\rho g w R \int_0^{\pi/2} (L - R\sin\theta)\cos\left(\frac{\pi}{2} - \theta\right) d\theta$$

$$\text{Note}: \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

$$= -\rho g w R \int_0^{\pi/2} (L\sin\theta - R\sin^2\theta) d\theta$$

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

$$= -\rho g w R \int_0^{\pi/2} \left[L\sin\theta - \frac{R}{2} + \frac{R}{2}\cos 2\theta \right] d\theta$$

$$= -\rho g w R \left[-L\cos\theta + \frac{R}{2}\theta + \frac{R}{4}\sin 2\theta \right]_0^{\pi/2}$$

$$= -\rho g w R \left[L\cos\theta - \frac{\pi R}{4} \right]$$

<u>Note:</u> Horizontal and vertical components of the resultant pressure force are both negative, so that they are acting in a direction opposite to x and y axis, respectively.

Line of action of vertical component

$$x' = \frac{1}{F_{R_v}} \int_A x p dA \cos(\frac{\pi}{2} - \theta)$$

$$\text{Note}: x = R\cos\theta$$

$$= \frac{\rho g}{F_{R_v}} \int_0^{\pi/2} (R\cos\theta)(L - R\sin\theta) x \cos(\frac{\pi}{2} - \theta) R d\theta$$

$$= \frac{\rho g w R}{F_{R_v}} \int_0^{\pi/2} (L\sin\theta\cos\theta - R\sin^2\theta\cos\theta) d\theta$$

$$= \frac{\rho g w R}{F_{R_v}} \left[(\frac{L\sin^2\theta}{2} - \frac{R\sin^3\theta}{3} \right]_0^{\pi/2}$$

$$= \frac{R(L/2 - L/3)}{(L - \pi R/4)}$$

$$y' = \frac{1}{F_{R_H}} \int_A ypdA\cos\theta$$

$$= \frac{\rho g w}{F_{R_H}} \int_0^{\pi/2} (R\sin\theta)(L - R\sin\theta)\cos\theta d\theta$$

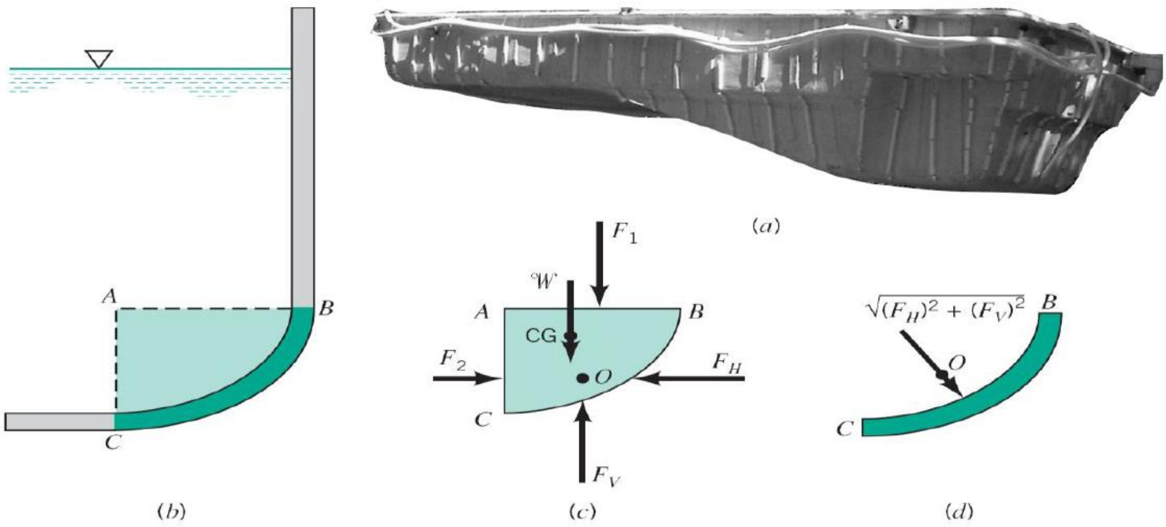
$$= \frac{\rho g w R}{F_{R_H}} \int_0^{\pi/2} (L\sin\theta\cos\theta - R\sin^2\theta\cos\theta)d\theta$$

$$= \frac{\rho g w R}{F_{R_V}} \left[(\frac{L\sin^2\theta}{2} - \frac{R\sin^3\theta}{3} \right]_0^{\pi/2}$$

$$= \frac{R(L/2 - L/3)}{(L - R/2)}$$

ALTERNATIVE APPROACH FOR CALCULATION OF RESULTANT FORCE ACTING ON CURVED SURFACES

The resultant fluid force acting on a curved submerged surface can be determined by integration as in the above example. This is generally a rather tedious process, and no simple general formulas can be developed. **As an alternative approach we will consider the equilibrium of the fluid volume enclosed by the curved surface of interest and the horizontal and vertical projections of this surface.**



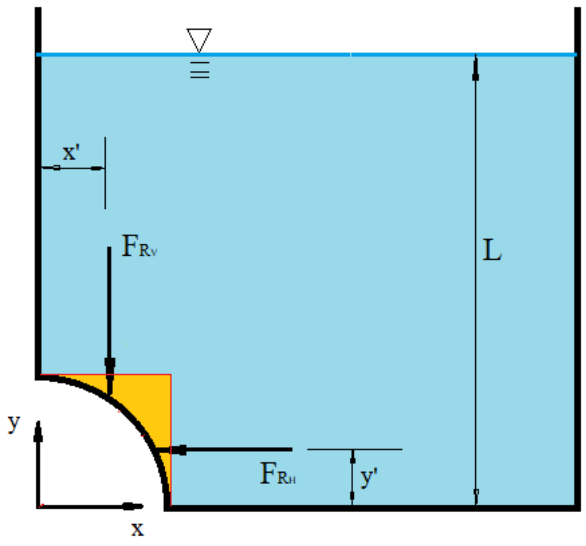
Consider the **section *BC*** shown in the figure above. This section has a unit length perpendicular to the plane of the paper.

- We **first isolate a volume of fluid** that is bounded by the surface of interest, in this instance section *BC*, and the horizontal plane surface *AB* and the vertical plane surface *AC*.
- ***Draw the* free-body diagram** for this volume as shown in Fig. *c*.
- The magnitude and location of forces $F1$ and $F2$ can be determined from the relationships for planar surfaces.
- The weight, **W**, is simply weight of the fluid in the enclosed volume.
- Forces ***FH*** and ***FV*** represent the components of the force that the tank *exerts on the fluid.*
- Fr*om the force balance, we can obtain FH and FV as follow:*

$$F_H = F_2 \qquad\qquad F_V = F_1 + W$$

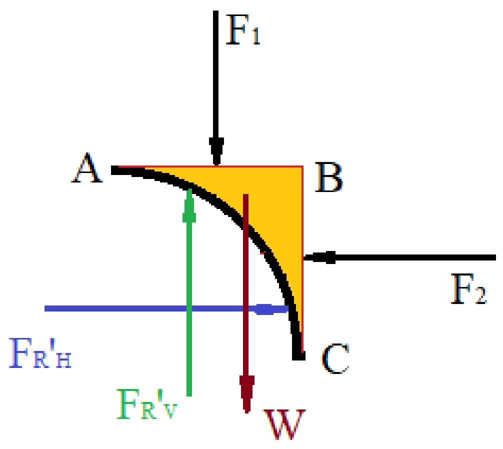
The resultant force of the fluid acting *on the curved surface BC* is equal and opposite in direction to that obtained from the free-body diagram.

Example: Solve the previous example using the second method.



Solution:

Draw the free body diagram of the isolated liquid.



From the free-body diagram

$F'_{R_H} = F_2 , \qquad F'_{R_V} = F_1 + W$

$F_{R_H} = -F'_{R_H} , \quad F_{R_V} = -F'_{R_V}$

W: weight of the isolated liquid

F_1: the hydrostatic force acting on surface AB

F_2: the hydrostatic force acting on surface BC

F'_{R_V} : the vertical component of the force exerted by curved surface AC.

F'_{R_H} : the horizontal component of the force exerted by curved surface AC.

$$F'_{R_H} = F_2 = p_c A_{BC} = \rho g h_c A_{BC} = \rho g \left(L - \frac{R}{2} \right) Rw = \rho g w R \left(L - \frac{R}{2} \right)$$

$$F'_{R_V} = F_1 + W = p_c A_{AB} + \rho g \forall = \rho g h_c A_{AB} + \rho g \left(R^2 w - \frac{1}{4} \pi R^2 w \right)$$

$$= \rho g (L - R) Rw + \rho g R w \left(R - \frac{1}{4} \pi R \right)$$

$$= \rho g R w \left(L - R + R - \frac{1}{4} \pi R \right) = \rho g R w \left(L - \frac{\pi R}{4} \right)$$

$$\therefore F_{R_H} = -F'_{R_H} = -\rho g w R \left(L - \frac{R}{2} \right)$$

$$F_{R_V} = -F'_{R_V} = -\rho g R w\left(L - \frac{\pi R}{4}\right)$$

$$\left.\begin{aligned} x' &= \frac{1}{F_{R_V}}\int_A xpdA\cos(\frac{\pi}{2}-\theta) \\ y' &= \frac{1}{F_{R_H}}\int_A ypdA\cos\theta \end{aligned}\right\}\text{same as calculated in the previous example.}$$

BUOYANCY

When a body is either fully or partially submerged in a fluid, a net force called the buoyant force acts on the body. This force is caused by the difference the pressure on the upper and lower surface of body. Consider the object shown in the figure immersed in a static fluid. We want to calculate the net vertical force that pressure exerts on the body.

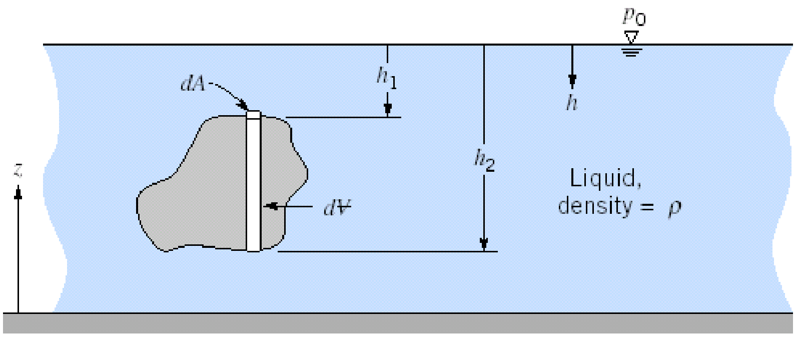


Fig. 3.9 Immersed body in static liquid.

$$dF_z = (p_0 + \rho g h_2)dA - (p_0 + \rho g h_1)dA = \rho g \underbrace{(h_2 - h_1)dA}_{d\forall} = \rho g d\forall$$

Thus the net vertical force on the body is

$$F_z = \int dF_z = \int_{\forall} \rho g d\forall = \rho g \forall$$

where $\forall$ is the volume of the object.

Thus the net vertical pressure force, or buoyancy force, equals the force of gravity on the liquid displaced by the object. This relation was reportedly used by Archimedes in 220 B.C., it is often called '**Archimedes Principle**'.

The line of action of the buoyancy force may be found using the methods that used in the previous section.

$$X_B = \frac{1}{F_B} \int x dF = \frac{1}{\rho g \forall} \int_{\forall} x d\forall$$

<u>Note:</u> The line of action of the buoyant force passes through the centroid of the displaced volume. This centroid is called the **center of buoyancy**.

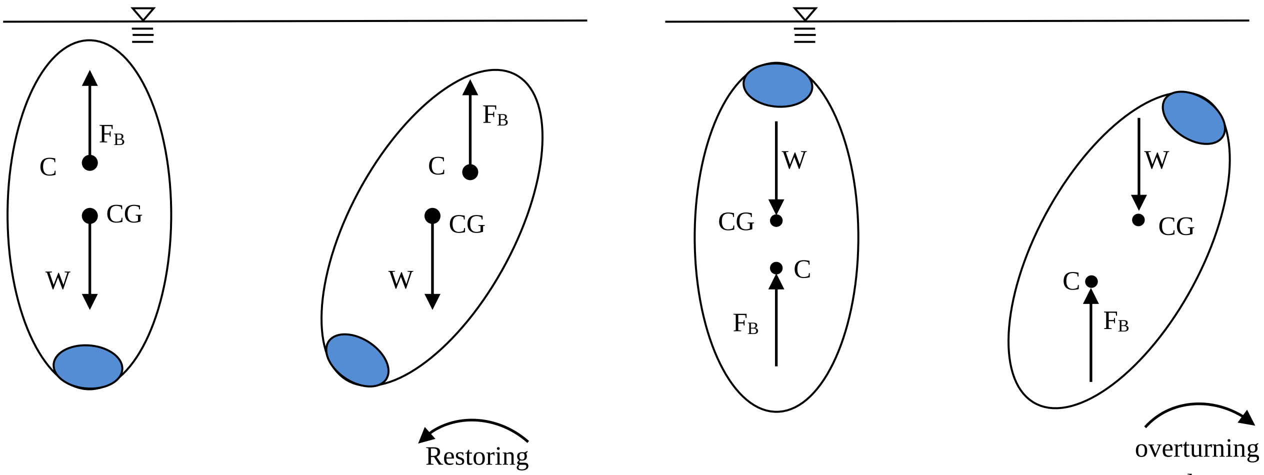
<u>Stability of Submerged and Floating Bodies</u>

The location of the line of action of the buoyancy force and the line of action of the force due to gravity determines the stability.

STABILITY

The location of the line of action of the buoyancy force and the line of action of the force due to gravity determines the stability.

Stability of a Completely Immersed Body

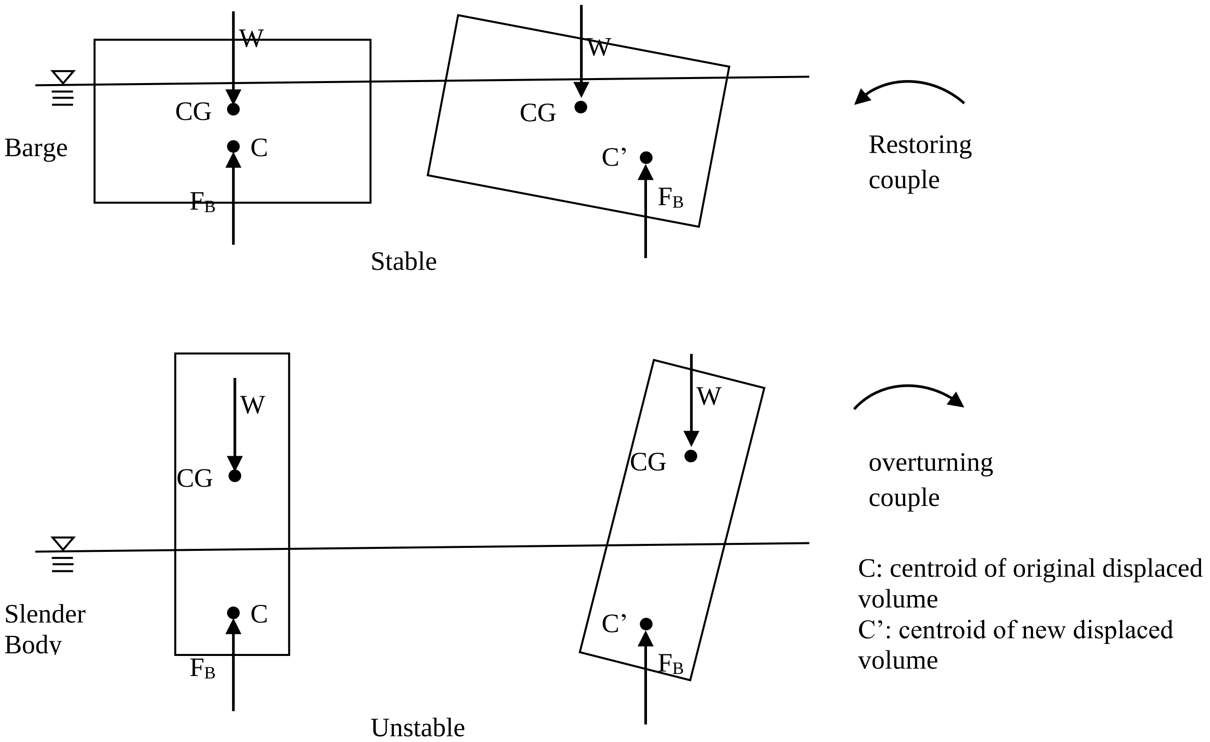


Stable

Center of gravity below centroid

Unstable

Center of gravity above centroid


W
CG
C
F_B
Barge
W
CG
C’
F_B
Restoring couple
Stable
W
CG
C
F_B
Slender Body
W
CG
C’
F_B
overturning couple
Unstable
C: centroid of original displaced volume
C’: centroid of new displaced volume

FLUIDS IN RIGID BODY MOTION

A fluid in rigid body motion moves without deformation as though it were a solid body. Since there is no deformation, there can be no shear stress. Consequently, the only surface stress on each element of fluid is that due to pressure. Hence, as in the case of static fluid, the force acting on a fluid element in rigid body motion is

$$d\vec{F} = (-grad\ p + \rho g)d\forall$$

or force on a fluid element of unit volume

$$\frac{d\vec{F}}{d\forall} = -grad\ p + \rho g$$

Using Newton's second law, we can write

$$d\vec{F} = \vec{a}dm$$

$$-\,grad\ p + \rho g = \rho\vec{a}$$

The physical significance f each term in this equation is

$$
\begin{array}{ccccccc}
-\,grad\ p & + & \rho g & = & & \rho\vec{a} & \\
\left\{\begin{array}{l} pressure\ force \\ per\,unit\,volume \\ at\,a\ point \end{array}\right\} & + & \left\{\begin{array}{l} body\ force \\ per\,unit\,volume \\ at\,a\ point \end{array}\right\} & = & \left\{\begin{array}{l} mass \\ per\,unit\,volume \\ \ \end{array}\right\} & \times & \left\{\begin{array}{l} acceleration \\ of\ \ fluid \\ particle \end{array}\right\}
\end{array}
$$

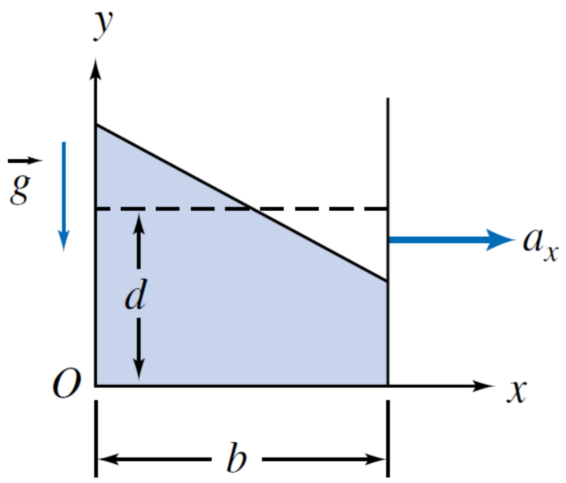
From the above vector equation, following scalar equations can be written

$$-\frac{\partial p}{\partial x} + \rho g_x = \rho a_x$$

$$-\frac{\partial p}{\partial y} + \rho g_y = \rho a_y$$

$$-\frac{\partial p}{\partial z} + \rho g_z = \rho a_z$$

<u>Example:</u> An open tank is used to transport liquid. What should be the maximum height of the liquid in tank to be sure that it will not spill over during the trip?



d=?

Basic equation

$$-\,grad\,P + \rho g = \rho \vec{a}$$

$$\left.\begin{array}{l} a_y = 0 \\ a_z = 0 \end{array}\right. \qquad \left.\begin{array}{l} g_z = 0 \\ g_x = 0 \\ g_y = -g \end{array}\right\} \Rightarrow \left.\begin{array}{l} \dfrac{\partial p}{\partial z} = 0 \\ \dfrac{\partial p}{\partial x} = -\rho a_x \\ \dfrac{\partial p}{\partial y} = -\rho g \end{array}\right\} \Rightarrow p = p(x,y)$$

Then the total change in pressure with change in x and y with dx and dy, can be written as

$$\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy = 0$$

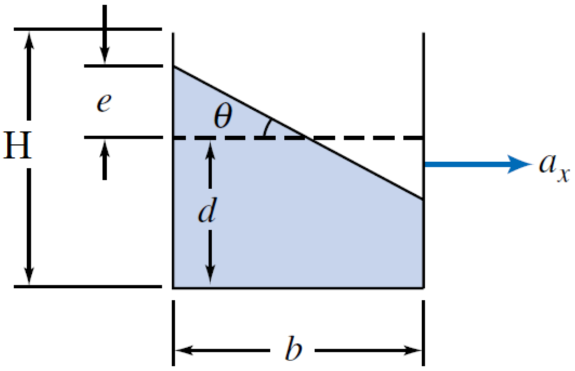
Since the free surface is open to atmosphere, the pressure is equal to atmospheric pressure and it is constant. Thus

$$dp = 0$$

$$\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy = 0$$

$$-\rho a_x dx - \rho g dy = 0 \quad \Rightarrow \left.\frac{dy}{dx}\right)_{free\,surface} = -\frac{a_x}{g} \qquad \Rightarrow \text{free surface is a straight line}$$

From the figure



$$\tan\theta = \frac{e}{\frac{b}{2}}$$

$$\left.\begin{aligned} e &= \frac{b}{2}\tan\theta \\ \tan\theta &= -\frac{dy}{dx} \\ \frac{dy}{dx} &= -\frac{a_x}{g} \end{aligned}\right\} \Rightarrow e = \frac{b}{2}\frac{a_x}{g}$$

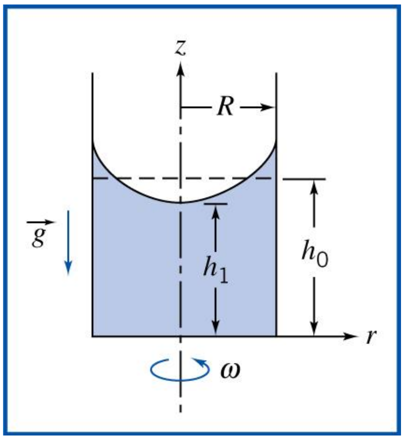
$$d = H - e$$

$$d = H - \frac{b}{2}\frac{a_x}{g} \quad \Leftarrow \text{maximum allowable liquid height}$$

FLUID ROTATING ABOUT A VERTICAL AXIS

A cylindrical container, partially filled with liquid, is rotated at a constant angular velocity ω, about its axis.

After a short time there is no relative motion; the liquid rotates with the cylinder as if the system were a rigid body. Determine the shape of the free surface.



Since there is a circumferential symmetry, the pressure is not function of θ. Then,

$$p=p(r,z)$$

$$\therefore\ dp=\left.\frac{\partial p}{\partial r}\right)_z dr+\left.\frac{\partial p}{\partial z}\right)_r dz \qquad (1)$$

In order to obtain pressure distribution, we need to find expression for $\left.\frac{\partial p}{\partial r}\right)_z$ and $\left.\frac{\partial p}{\partial z}\right)_r$. This can be obtained by writing Newton's second law in z and r directions (or writing equation $-\,grad\ p+\rho g=\rho\vec{a}$ in cylindrical coordinate system).

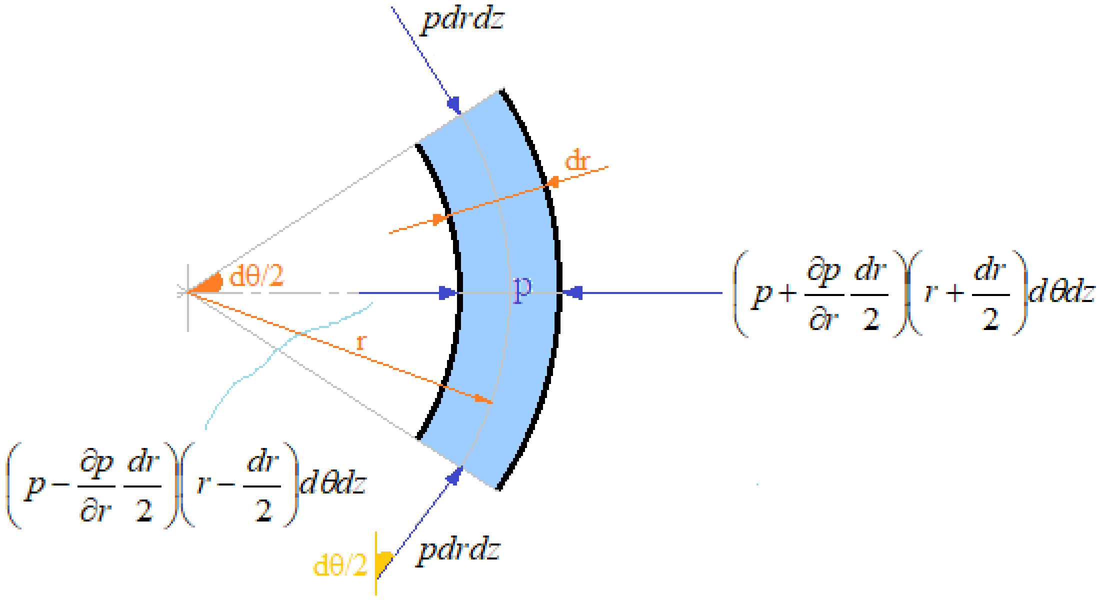
From equation

$$\left.\begin{array}{l} -\left.\dfrac{\partial p}{\partial z}\right)_r+\rho g_z=\rho a_z \\ g_z=-g \\ a_z=0 \end{array}\right\}\Rightarrow -\left.\frac{\partial p}{\partial z}\right)_r=-\rho g \qquad (2)$$

Similarly,

$$\left.\begin{array}{l} -\dfrac{\partial p}{\partial r}+\rho g_r=\rho a_r \\ g_r=0 \\ a_r=-\omega^2 r \end{array}\right\}\Rightarrow \frac{\partial p}{\partial r}=\rho\omega^2 r \qquad (3)$$

The same expression can also be obtained by applying Newton's second law in the r-direction to a suitable differential element.



The pressure at the center of the element is P. Using a Taylor series expansion, we express forces acting in the rθ plane on the element as shown in the figure.

Writing Newton's second law in the r-direction, we have

$$\begin{aligned}\sum dF_r \bullet &= a_r dm \\ &= a_r \rho d\forall \\ &= -\omega^2 r \rho d\forall \\ &= -\omega^2 r \rho r d\theta dr dz\end{aligned}$$

From the figure

$$\sum dF_r = \left(p-\frac{\partial p}{\partial r}\frac{dr}{2}\right)\left(r-\frac{dr}{2}\right)d\theta dz - \left(p+\frac{\partial p}{\partial r}\frac{dr}{2}\right)\left(r+\frac{dr}{2}\right)d\theta dz + 2pdrdz\sin\frac{d\theta}{2}$$

Expanding and canceling like terms, recognizing $\sin\frac{d\theta}{2}=\frac{d\theta}{2}$ gives

$$\sum dF_r = d\theta dz\left\{pr - p\frac{dr}{2} - r\frac{\partial p}{\partial r}\frac{dr}{2} + \frac{\partial p}{\partial r}\frac{dr}{2} - pr - p\frac{dr}{2} - r\frac{\partial p}{\partial r}\frac{dr}{2} - \frac{\partial p}{\partial r}\left(\frac{dr}{2}\right)^2 + pdr\right\}$$

$$\sum dF_r = d\theta dz\left\{-r\frac{\partial p}{\partial r}dr\right\}$$

Then

$$-r\frac{\partial p}{\partial r}drd\theta dz = -\omega^2 r\rho rd\theta drdz$$

$$\frac{\partial p}{\partial r} = -\omega^2 r \qquad (3)$$

Substituting (2) and (3) into (1), we get

$$dp = \rho\omega^2 rdr - \rho gdz$$

To obtain the pressure difference between a reference point (r_1,z_1), where the pressure is P_1, and arbitrary point (r,z), where the pressure is P, we must integrate

$$\int_{p_1}^{p} dp = \int_{r_1}^{r} \rho\omega^2 rdr - \int_{p_1}^{p} \rho gdz$$

$$(p - p_1) = \frac{\rho\omega^2}{2}(r^2 - r_1^2) - \rho g(z - z_1)$$

Taking the reference point on the cylinder axis at the free surface gives

$$p_1 = p_{atm}, \quad r_1 = 0, \quad z_1 = h_1$$

Then

$$p - p_{atm} = \frac{\rho\omega^2 r^2}{2} - \rho g(z - h_1)$$

Since the free surface is a surface of constant pressure (p=p_{atm}), the equation of the free surface is given by

$$0 = \frac{\rho\omega^2 r^2}{2} - \rho g(z - h_1) \;\Rightarrow\; z = h_1 + \frac{(\omega r)^2}{2g} \;\Rightarrow\; \text{Equation of the free surface.}$$

(parabola with vertex on the axis at $z = h_1$)

We can solve for the height h_1 in terms of the original height h_o and R. To do this, we use the fact that the volume of the fluid must remain constant.

with no rotation $\quad \forall = \pi R^2 h_0$

with rotation $$\forall = \int_0^R \int_0^{rz} 2\pi r dz dr = \int_0^R 2\pi r z dr$$

$$\forall = \int_0^R 2\pi\left(h_1 + \frac{\omega^2 r^2}{2g}\right) r dr$$

$$\forall = 2\pi\left[h_1 \frac{r^2}{2} + \frac{\omega^2 r^4}{8g}\right]_0^R = \pi\left[h_1 R^2 + \frac{\omega^2 R^4}{4g}\right]$$

Then equating these two expression for volume,

$$\pi R^2 h_0 = \pi\left[h_1 R^2 + \frac{\omega^2 R^4}{4g}\right]$$

$$\Rightarrow \quad \boxed{h_1 = h_0 - \frac{(\omega R)^2}{4g}}$$

Finally,

$$z = h_0 - \frac{(\omega R)^2}{4g} - \frac{(\omega r)^2}{2g}$$

$$\boxed{z = h_0 - \frac{(\omega R)^2}{2g}\left[\frac{1}{2} - \left(\frac{r}{R}\right)^2\right]}$$ Equation of the free surface

FUNDAMENTAL CONCEPTS FOR FLOW ANALYSIS

We covered methods of analysis of nonflowing fluids in the previous chapter. In this chapter, we develop the fundamental concepts of flow analysis, including way to describe fluid flow, natural laws that govern fluid flow, different approaches to formulating mathematical models of fluid flow, and methods that engineers use to flow problems.

The Fundamental Laws

Experience have shown that all fluid motion analysis must be consistent with the following fundamental laws of **nature.**

- The law of conservation of mass. Mass can be neither created nor destroyed. It can only be transported or stored.
- Newton's three law of motion:
    1. A mass remains in a state of equilibrium, that is, at rest or moving at constant velocity, unless acted on by unbalanced force.
    2. The rate of change of momentum of mass is equal to the net force acting on the mass.
    3. Any force action has an equal (in magnitude) and opposite (in direction) force reaction.
- The first law of thermodynamics (law of conservation of energy) Energy, like mass, can be neither created nor destroyed. Energy can be transported, changed in form, or stored.
- The second law of thermodynamics: The entropy of the universe must increase or, in the ideal case, remain constant in all natural processes.
- The state of postulate (law of property relations): The various properties of a fluid are related. If a certain minimum number (usually two) of fluid's properties are specified, the remainder of the properties can be determined.

NOTE: These laws apply to all flows. They do not depend on the nature of the fluid, the geometry of the boundaries, or anything else. As far as we know, they have always been true and will continue to be true unless they are suspended by the creator of the universe. Hence, we can firmly base analysis of all flows on these laws.

Constitutive Relations

In addition to these universal laws, several less fundamental laws, such as Newton's law of viscosity, Fourier's law of conduction, are needed to solve flow problems.

These laws are true for some fluids.

Mathematical Formulation

The fundamental laws are the basis of our understanding of fluid motion. However, besides understanding, an engineer needs to know qualitatively the velocity, and the pressure to calculate the effects of the fluid on surfaces that it contacts, such as force exerted by the fluid on a surface, pressure drop in a pipe flow, etc.

To obtain predictive capability, the fundamental laws must be expressed mathematically and they must be solved to predict velocity or pressure.

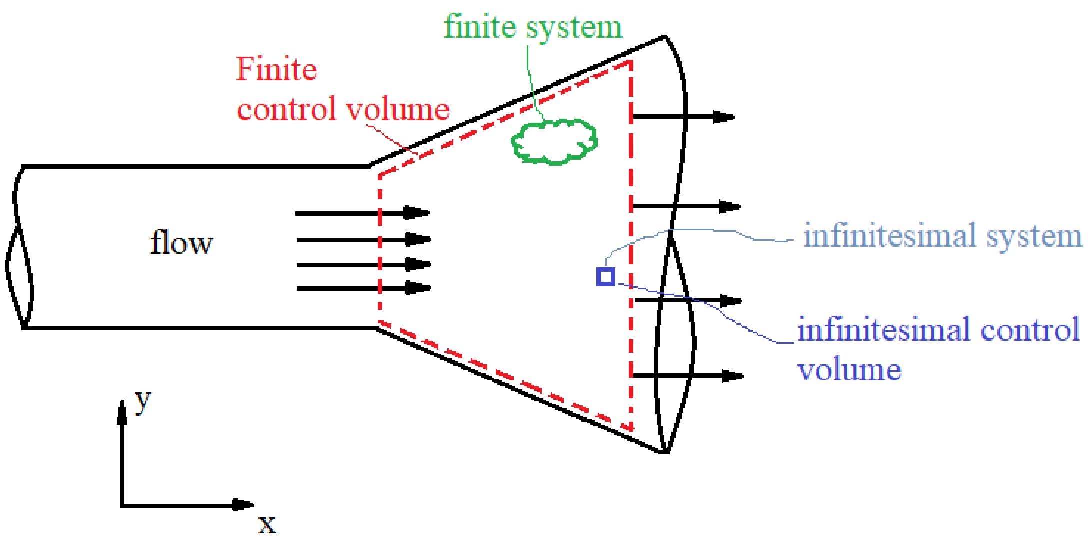
To formulate the fundamental laws, we choose both a point of view and a mathematical method.

System versus Control Volume

We may apply the fundamental laws to either a system or a control volume.

System : a specific fluid mass selected for analysis.
Control Volume : a specific region of space selected for analysis.

System and control volume may be either infinitesimally small or finite.



The system of point of view is related to a Lagrangian description of flow. Its advantages is that all the fundamental laws may be expressed directly in terms of a specific collection of mass.

Control volume point of view is related to an Eulerian description of flow. Its advantage is that control volumes are easier to use for problem solution.

Thus we adopt the system point of view to formulate the fundamental laws, but use the control volume point of view to apply them to problems. Fortunately, we can formally connect the two points view by purely mathematical relationships.

<u>Differential versus Integral Formulation</u>

We must now consider the level of detail of the resulting flow analysis. We must choose between a detailed point by point description and a global or lumped description.

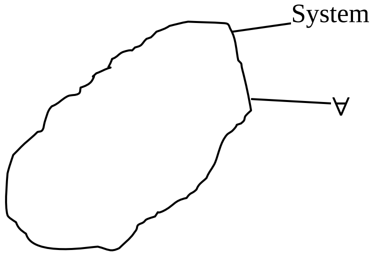
When a point by point (local) description is desired, fundamental laws are applied to an infinitesimal control volume. The result will be a set of differential equations with the fluid velocity and pressure as dependent variables and the location (x, y, z) and time as independent variables. Solution of these differential equations, together with boundary conditions, will be two function V(x, y, z, t), and P(x, y, z, t) that can tell us velocity the velocity and pressure at every point.

When global information such as flow rate, force and temperature change between inlet and outlet is desired, the fundamental laws are applied to a finite control volume.

The result will be a set of integral equations.

BASIC LAWS FOR A SYSTEM

Conservation of Mass



$$\left.\frac{dM}{dt}\right|_{system} = 0$$

where

$$M_{sys} = \int_{\substack{mass\\(sys)}} dm = \int_{\forall\, sys} \rho d\forall$$

Newton's Second Law

$$\vec{F} = \left.\frac{d\vec{P}}{dt}\right|_{sys}$$

$$\vec{P} : linear\ momentum$$

$$\vec{P} = \int_{\substack{mass\\(sys)}} \vec{V} dm = \int_{\forall\, sys} \vec{V} \rho d\forall$$

The First Law of Thermodynamics

$$\delta Q - \delta W = dE$$

in the rate form

$$\dot{Q} - \dot{W} = \left.\frac{dE}{dt}\right|_{sys}$$

The total energy of the system is given by

$$E_{sys} = \int_{\substack{mass\\(sys)}} e dm = \int_{\forall\, sys} e \rho d\forall$$

$$e = u + \frac{v^2}{2} + gz$$

The Second Law of Thermodynamics

If an amount of heat δQ is transferred to a system at temperature T, the second law of thermodynamics states that the change in entropy *ds* of the system is given by

$$ds \geq \frac{\delta Q}{T}$$

on the rate basis

$$\left.\frac{ds}{dt}\right|_{sys} \geq \frac{1}{T}\dot{Q}$$

Total entropy of the system is

$$S_{sys} = \int_{\substack{mass \\ (sys)}} s\,dm = \int_{\substack{\forall \\ (sys)}} \rho s\,d\forall$$

RELATION OF SYSTEM DERIVATIVES TO THE CONTROL VOLUME FORMULATION

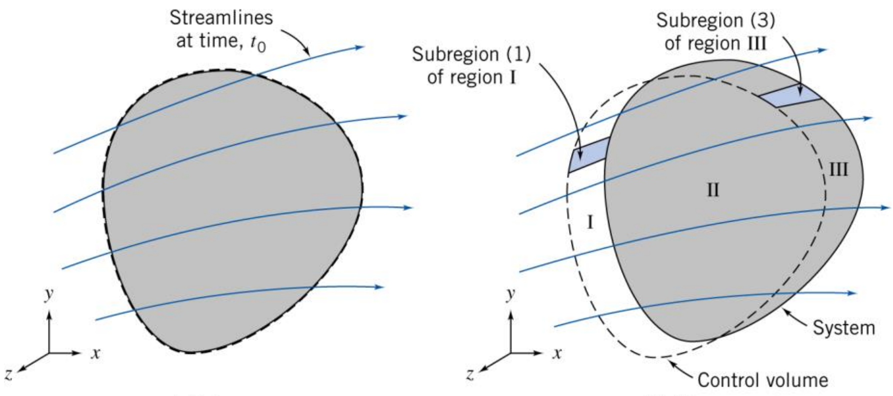
The above equations involve the time derivative of an extensive property of the system (mass, momentum, energy, entropy). All the above equations can be expressed in terms of a general intensive property η. Thus

$$N_{sys} = \int_{\substack{mass \\ (sys)}} \eta\,dm = \int_{\substack{\forall \\ (sys)}} \eta\rho\,d\forall$$

Comparing this with the above equations, we see that when

N=M	then	η=1	N=E	then	η=e
N=$\vec{P}$	then	η=$\vec{V}$	N=S	then	η=s

Consider a system and control volume whose boundaries coincide at t_0.



(a) Time, t_0 (b) Time, $t_0 + \Delta t$

Objective: To relate the $\left.\frac{dN}{dt}\right|_{system}$ to the time variations of this property (N) associated with the control volume.

From the definition of a derivative,

$$\left.\frac{dN}{dt}\right|_{system} \equiv \lim_{\Delta t \to 0} \frac{\left.N_s\right)_{t_0+\Delta t} - \left.N_s\right)_{t_0}}{\Delta t} \tag{1}$$

At $t+\Delta t$, the system occupies regions II and III, at t_0, the system and the control volume coincide, we can write

$$\left.N_s\right)_{t_0} = \left(N_{C\forall}\right)_{t_0} = \int_{C\forall} \eta\rho d\forall \tag{2}$$

$$\left.N_s\right)_{t_0+\Delta t} = \left(N_{II} + N_{III}\right)_{t_0+\Delta t} = \left(N_{C\forall} - N_I + N_{III}\right)_{t_0+\Delta t}$$

$$\left.N_s\right)_{t_0+\Delta t} = \left[\int_{C\forall} \eta\rho d\forall\right]_{t_0+\Delta t} - \left[\int_{I} \eta\rho d\forall\right]_{t_0+\Delta t} + \left[\int_{III} \eta\rho d\forall\right]_{t_0+\Delta t} \tag{3}$$

Substituting these expressions şnto the definition of the system derivative

$$\left.\frac{dN}{dt}\right)_{system} = \lim_{\Delta t \to 0} \frac{\left[\int_{C\forall} \eta\rho d\forall\right]_{t_0+\Delta t} - \left[\int_{I} \eta\rho d\forall\right]_{t_0+\Delta t} + \left[\int_{III} \eta\rho d\forall\right]_{t_0+\Delta t} - \left[\int_{C\forall} \eta\rho d\forall\right]_{t_0}}{\Delta t}$$

or

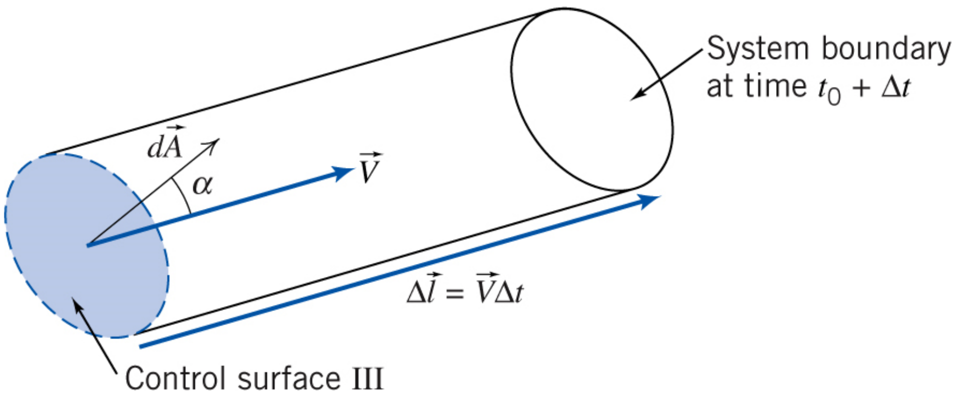
$$\left.\frac{dN}{dt}\right)_{system} = \underbrace{\lim_{\Delta t\to 0}\frac{\left[\int_{C\forall}\eta\rho d\forall\right]_{t_0+\Delta t}-\left[\int_{C\forall}\eta\rho d\forall\right]_{t_0}}{\Delta t}}_{1} + \underbrace{\lim_{\Delta t\to 0}\frac{\left[\int_{III}\eta\rho d\forall\right]_{t_0+\Delta t}}{\Delta t}}_{2} - \underbrace{\lim_{\Delta t\to 0}\frac{\left[\int_{I}\eta\rho d\forall\right]_{t_0+\Delta t}}{\Delta t}}_{3} \tag{4}$$

Term 1 in Eq. 4 simplifies to

$$\lim_{\Delta t\to 0}\frac{\left[\int_{C\forall}\eta\rho d\forall\right]_{t_0+\Delta t}-\left[\int_{C\forall}\eta\rho d\forall\right]_{t_0}}{\Delta t} = \frac{\partial}{\partial t}\int\eta\rho d\forall \tag{5}$$

$$\lim_{\Delta t\to 0}\frac{\left[\int_{III}\eta\rho d\forall\right]_{t_0+\Delta t}}{\Delta t} = \lim_{\Delta t\to 0}\frac{\left.N_{III}\right)_{t_0+\Delta t}}{\Delta t}$$

To evaluate $\left.N_{III}\right)_{t_0+\Delta t}$ let us look at an enlarged view of a typical subregion of region III.



$$\left.\begin{array}{l}\left.N_{III}\right)_{t_0+\Delta t} = \left[\int_{III}\eta\rho d\forall\right]_{t_0+\Delta t}\\ d\forall = \Delta l\cos\alpha dA\end{array}\right\} \Rightarrow \left.N_{III}\right)_{t_0+\Delta t} = \left[\int_{CS_{III}}\eta\rho\Delta l\cos\alpha dA\right]_{t_0+\Delta t}$$

Note: the angle α will always be less than $\pi/2$ over the entire area of the control surface bounding region III.

In the above expression, *Δl* is the distance travelled by a particle on the system surface during the interval *Δt* along the streamline that existed at *t0*.

$$\lim_{\Delta t\to 0}\frac{\left[\int_{III}\eta\rho d\forall\right]_{t_0+\Delta t}}{\Delta t}=\lim_{\Delta t\to 0}\int_{CS_{III}}\eta\rho\frac{\Delta l}{\Delta t}\cos\alpha dA$$

Note: $\lim_{\Delta t\to 0}\frac{\Delta l}{\Delta t}=|\vec{V}|$ and $dA=|d\vec{A}|$

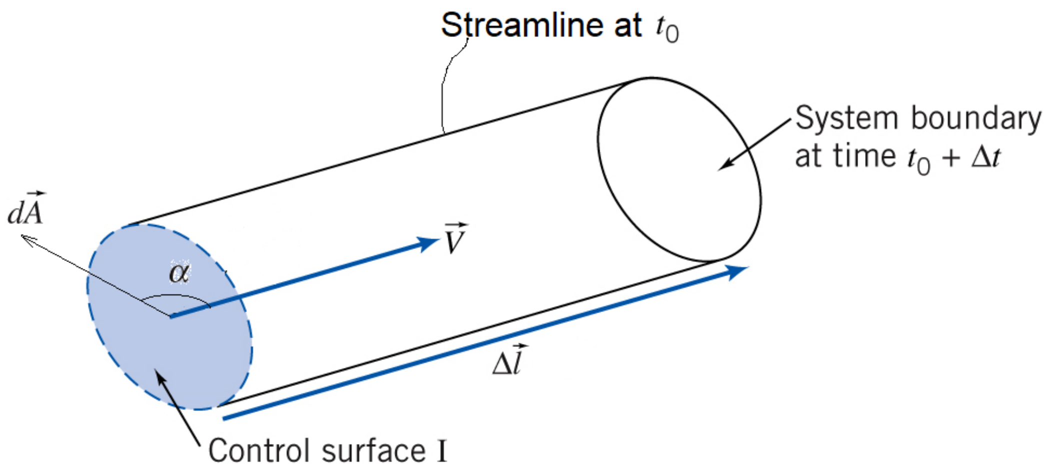
Hence

$$\lim_{\Delta t\to 0}\frac{\left[\int_{III}\eta\rho d\forall\right]_{t_0+\Delta t}}{\Delta t}=\int_{CS_{III}}\eta\rho|\vec{V}|\cos\alpha|d\vec{A}| \qquad (6)$$

The term 3 in Eq. 4 simplifies to

$$-\lim_{\Delta t\to 0}\frac{\left[\int_{I}\eta\rho d\forall\right]_{t_0+\Delta t}}{\Delta t}=\lim_{\Delta t\to 0}\frac{\left.N_I\right)_{t_0+\Delta t}}{\Delta t}$$

To evaluate $\left.N_I\right)_{t_0+\Delta t}$, look at an enlarged view of a typical subregion



$$d\forall=\Delta l(-\cos\alpha)dA$$

$$-\lim_{\Delta t\to 0}\frac{\left[\int_{I}\eta\rho d\forall\right]_{t_0+\Delta t}}{\Delta t}=-\lim_{\Delta t\to 0}\frac{\int_{CS_I}\eta\rho\Delta l\cos\alpha dA}{\Delta t}$$
$$=+\lim_{\Delta t\to 0}\int_{CS_I}\eta\rho\frac{\Delta l}{\Delta t}\cos\alpha dA=+\int_{CS_I}\eta\rho|\vec{V}|\cos\alpha|d\vec{A}| \qquad (7)$$

Substituting Eqs. (5),(6) and (7) into (4)

$$\left.\frac{dN}{dt}\right)_{sys} = \frac{\partial}{\partial t}\int_{C\forall} \eta\rho d\forall + \int_{CS_{III}} \eta\rho\left|\vec{V}\right|\cos\alpha\left|d\vec{A}\right| + \int_{CS_I} \eta\rho\left|\vec{V}\right|\cos\alpha\left|d\vec{A}\right|$$

$$CS = CS_I + CS_{III}$$

Hence we can write,

$$\left.\frac{dN}{dt}\right)_{sys} = \frac{\partial}{\partial t}\int_{C\forall} \eta\rho d\forall + \int_{CS} \eta\rho\left|\vec{V}\right|\cos\alpha\left|d\vec{A}\right|$$

Recognizing that $\left|\vec{V}\right|\cos\alpha\left|d\vec{A}\right| = \vec{V}\cdot d\vec{A}$

$$\left.\frac{dN}{dt}\right)_{sys} = \frac{\partial}{\partial t}\int_{C\forall} \eta\rho d\forall + \int_{CS} \eta\rho\vec{V}\cdot d\vec{A}$$

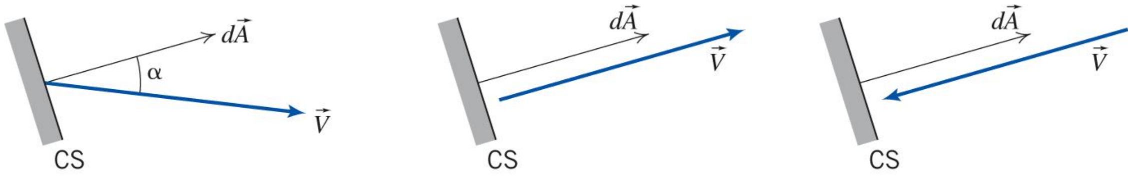
It is important to recall that in deriving the above equation, the limiting process (taking the limit as $\Delta t \to 0$) ensured that the relation is valid at the instant when the system and control volume coincide.

$\left.\frac{dN}{dt}\right)_{sys}$: the total rate of change of any arbitrary extensive property of the system.

$\frac{\partial}{\partial t}\int_{C\forall} \eta\rho d\forall$: the time rate of change of the arbitrary extensive property N within the control volume

$\int_{CS} \eta\rho\vec{V}\cdot d\vec{A}$: the net rate of flux of the extensive property N through the control surface.

Evaluating the scalar product



$\vec{V}\cdot d\vec{A} = VdA\cos\alpha$
(*a*) General inlet/exit

$\vec{V}\cdot d\vec{A} = +VdA$
(*b*) Normal exit

$\vec{V}\cdot d\vec{A} = -VdA$
(*c*) Normal inlet

CONSERVATION OF MASS (Continuity Equation)

Combining the law of conservation of mass with the transport theorem yields one of the most useful equations in all fluid mechanics: the continuity equation.

Recall that conservation of mass states simply that the mass of a system is constant,

$$\left.\frac{dM}{dt}\right)_{sys} = 0, \qquad M_{sys} = \int_{\forall_{SYS}} \rho d\forall$$

The system and control volume formulation of the conservation of mass, we set

N=M then η=1

with this substitution, we obtain

$$\left.\frac{dM}{dt}\right)_{sys} = \frac{\partial}{\partial t}\int_{C\forall} \rho d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A} = 0$$

$$\frac{\partial}{\partial t}\int_{C\forall} \rho d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A} = 0$$ Continuity equation for a finite control volume

$\frac{\partial}{\partial t}\int_{C\forall} \rho d\forall$ rate of change of mass within the control volume

$\int_{CS} \rho \vec{V} \cdot d\vec{A}$ net rate of flus through the control surface

NOTE: $\vec{V}$ is the velocity measured relative to the control surface. The sign of the dot product $\rho \vec{V} \cdot d\vec{A}$ depends on the direction of velocity vector $\vec{V}$, relative to the area vector $d\vec{A}$. $\rho \vec{V} \cdot d\vec{A}$is positive where flow is out through the control surface, negative where flow is in through the control surface, and zero where flow is tangent to surface.

Special Cases

1. Incompressible Flow:

For incompressible flow, ρ = constant

$$\frac{\partial}{\partial t}\rho\int_{C\forall} d\forall + \rho\int_{CS}\vec{V}\cdot d\vec{A} = 0$$

$$\frac{\partial}{\partial t}[\rho\forall] + \rho\int_{CS}\vec{V}\cdot d\vec{A} = 0 \qquad \Rightarrow \int_{CS}\vec{V}\cdot d\vec{A} = 0$$

$$\frac{\partial}{\partial t}[\rho\forall] = 0 \quad \begin{bmatrix}\rho = \text{constant} \\ \forall = \text{constant}\end{bmatrix}$$

$\int \vec{V}\cdot d\vec{A}$ is called the volume flow rate of flow over a section of the control surface.

2. Steady Flow

$$\frac{\partial}{\partial t} = 0$$

Hence the continuity equatşon becomes,

$$\int_{CS}\rho\vec{V}\cdot d\vec{A} = 0, \quad \text{[flow could be compressible]}$$

3. Uniform Flow

The velocity is constant across the entire area at a section when density is also constant at a section, then at section n

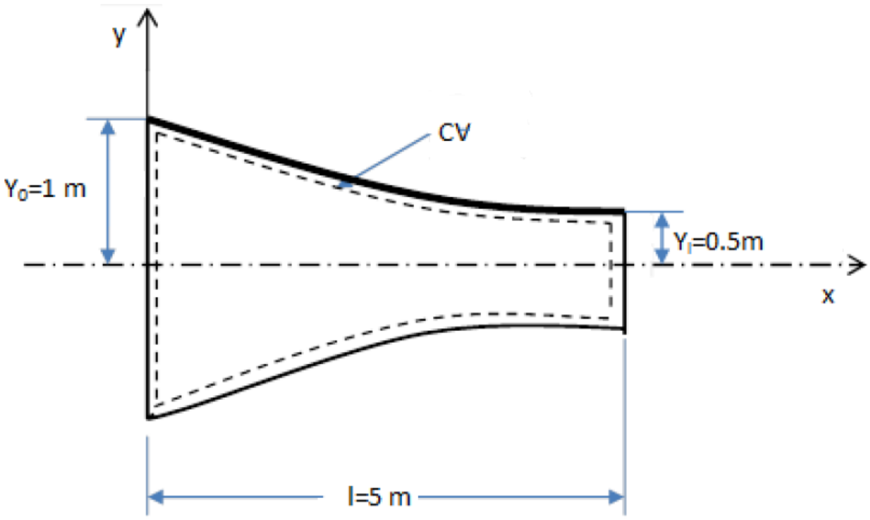
$$\int_{A_n}\rho\vec{V}\cdot d\vec{A} = \rho_n\vec{V}_n\cdot\vec{A}_n = \pm|\rho_n V_n A_n|$$

Example:

A constant density fluid flows in the converging, two-dimensional channel shown in the figure. The width perpendicular to the paper is quite large compared to the channel height. The velocity in the z-direction is zero. The channel half height y and the fluid velocity in x-direction are given by

$$y = \frac{y_0}{1 + x/l} \quad \text{and} \quad u = u_0\left(1 + \frac{x}{l}\right)\left[1 - \left(\frac{y}{Y}\right)^2\right]$$

where u_0=1.0 m/s



Show that the flow field satisfies the continuity equation.

General continuity equation

$$\frac{\partial}{\partial t}\int_{C\forall} \rho d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A} = 0$$

$$\frac{\partial}{\partial t}\int_{C\forall} \rho d\forall = 0 \quad [\rho = \text{constant}]$$

$$\int_{CS} \vec{V}\cdot d\vec{A} \overset{?}{=} 0$$

For the control volume shown in the figure, along the walls of the channel, *u=v=0*, hence

$$\int_{CS} \vec{V}\cdot d\vec{A} = \int_{A_{in}} \vec{V}\cdot d\vec{A} + \int_{A_{out}} \vec{V}\cdot d\vec{A} \overset{?}{=} 0$$

$$= -\int_{A_{in}} udA + \int_{A_{out}} udA \overset{?}{=} 0$$

$$-\int_{-Y_0}^{+Y_0} u_0\left(1+\frac{x}{l}\right)\left[1-\left(\frac{y}{Y_0}\right)^2\right] wdy + \int_{-Y_l}^{+Y_l} u_0\left(1+\frac{x}{l}\right)\left[1-\left(\frac{y}{Y_l}\right)^2\right] wdy \overset{?}{=} 0$$

$$-wu_0\left[y-\frac{y^3}{3Y_0^2}\right]_{-Y_0}^{+Y_0} + 2wu_0\left[y-\frac{y^3}{3Y_l^2}\right]_{-Y_l}^{+Y_l} \overset{?}{=} 0$$

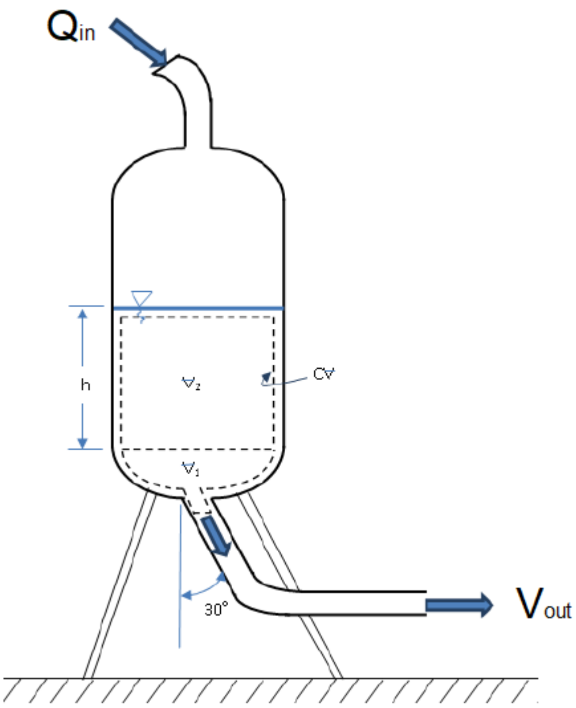
$$\left.\begin{array}{l} -\dfrac{4wu_0Y_0}{3} + \dfrac{8wu_0Y_l}{3} \overset{?}{=} 0 \\ u_0 = 1m/s,\ Y_0 = 1m, Y_l = 0.5m \end{array}\right\} \Rightarrow -\frac{4}{3}w + \frac{4}{3}w \overset{?}{=} 0$$

$$0 = 0$$

$\therefore$ Flow satisfies the continuity equation.

Example:

Water is being added to a storage tank at the rate of 2000 lt/min. At the same time, water flows through a 5 cm inside diameter pipe with an average velocity of 18 m/s. The storage tank has an inside diameter of 300 cm. Find the rate at which the water level rises or falls.



GIVEN

in flow rate 2000 lt/min.

storage tank diameter 300 cm

discharge pipe diameter 5 cm

discharge velocity 18 m/s

Basic Equation

Continuity equation

$$\frac{\partial}{\partial t}\int_{C\forall}\rho d\forall+\int_{CS}\rho\vec{V}\cdot d\vec{A}=0$$

$$\frac{\partial}{\partial t}\int_{C\forall}\rho d\forall=0\quad[\rho=\text{constant}]$$

Continuity equation,

$$\frac{\partial}{\partial t}\int_{C\forall}\rho d\forall+\int_{CS}\rho\vec{V}\cdot d\vec{A}=0$$

$$\rho\frac{\partial\forall}{\partial t}+\int_{A_{in}}\rho\vec{V}\cdot d\vec{A}+\int_{A_{out}}\rho\vec{V}\cdot d\vec{A}=0$$

$$\left.\begin{array}{l}\rho\dfrac{\partial\forall}{\partial t}+(-\underbrace{|\rho VA_{in}|}_{Q_{in}})+(+|\rho VA_{out}|)=0\\ \forall=\forall_1+\forall_2=\forall_1+A_T h\end{array}\right\}\Rightarrow\frac{d}{dt}(\forall_1+A_T h)-Q_{in}+(VA)_{out}=0$$

$$A_T\frac{dh}{dt}=Q_{in}-(VA)_{out}$$

$$\frac{dh}{dt}=\frac{Q_{in}-(VA)_{out}}{A_T}=\frac{\frac{2000\times10^{-3}}{60}\left[\frac{m^3}{s}\right]-18\left[\frac{m}{s}\right]\frac{\pi(0.05)^2}{4}\left[m^2\right]}{\frac{\pi}{4}3^2\left[m^2\right]}$$

$$\frac{dh}{dt}=-2.8\times10^{-4}\left[\frac{m}{s}\right]$$

MOMENTUM EQUATION FOR INERTIAL CONTROL VOLUME

In this section we will develop mathematical formulation of Newton's Second Law for an inertial control volume.

Inertial control volume is the control volumethat is not accelerating relative to a stationary frame of reference (inertial control volume).

Recall that Newton's second law for a system moving relative to an inertial coordinate system was

$$\vec{F} = \left.\frac{d\vec{P}}{dt}\right)_{sys} \quad \text{where} \quad \vec{P} = \int_{\forall_{sys}} \vec{V}\rho d\forall \quad \text{linear mometum, } \vec{F} \text{ total resultant force}$$

$$\vec{F} = \vec{F}_S + \vec{F}_B$$

Using the relation between the system and control volume formulations

$$\left.\frac{dN}{dt}\right)_{sys} = \frac{\partial}{\partial t}\int_{C\forall} \eta\rho d\forall + \int_{CS} \eta\rho\vec{V}\cdot d\vec{A}$$

and setting $N = \vec{P}$ and $\eta = \vec{V}$, we obtain

$$\left.\frac{d\vec{P}}{dt}\right)_{sys} = \frac{\partial}{\partial t}\int_{C\forall} \vec{V}\rho d\forall + \int_{CS} \vec{V}\rho\vec{V}\cdot d\vec{A}$$

Note:

$$\left.\frac{d\vec{P}}{dt}\right)_{sys} = \left.\vec{F}\right)_{on\ sys}$$

Since, in deriving the relation between the system and control volume formulation, the system and control volume coincided at to

$$\left.\vec{F}\right)_{on\ sys} = \left.\vec{F}\right)_{on\ control\ \ volume}$$

Hence, we can write,

$$\vec{F} = \vec{F}_S + \vec{F}_B = \frac{\partial}{\partial t} \int_{C\forall} \vec{V} \rho d\forall + \int_{CS} \vec{V} \rho \vec{V} \cdot d\vec{A}$$ **MOMETUM EQUATION**

This equation states that the sum of all forces (surface and body forces) acting on a nonaccelerating control volume is equal to the sum of the rate of the change of momentum inside the control volume and the net rate of efflux of momentum through the control surface.

$\vec{F}_S = \int_A - p\, d\vec{A}$ surface force due to pressure

$\vec{F}_B = \int_{C\forall} \rho\, \vec{g}\, d\vec{A}$ body force due to pressure

Sometimes surface force $\vec{F}_S$ may also include shear force.

The momentum equation is a vector equation. From this vector equation, a scalar component in each direction can be written, i.e.

$$\vec{F}_x = \vec{F}_{S_x} + \vec{F}_{B_x} = \frac{\partial}{\partial t} \int_{C\forall} u \rho d\forall + \int_{CS} u \rho \vec{V} \cdot d\vec{A}$$

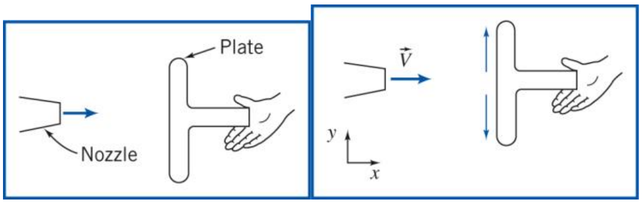
$$\vec{F}_y = \vec{F}_{S_y} + \vec{F}_{B_y} = \frac{\partial}{\partial t} \int_{C\forall} v \rho d\forall + \int_{CS} v \rho \vec{V} \cdot d\vec{A}$$

$$\vec{F}_z = \vec{F}_{S_z} + \vec{F}_{B_z} = \frac{\partial}{\partial t} \int_{C\forall} w \rho d\forall + \int_{CS} w \rho \vec{V} \cdot d\vec{A}$$

The momentum equation is usually used to calculate force interactions between a moving fluid and solid objects in contact with it.

Example:

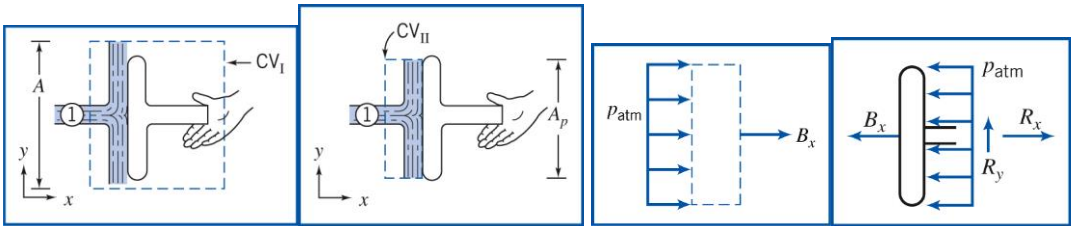
Water from a stationary nozzle strikes a flat plate as shown. The velocity of the water leaving the nozzle is 15 m/sec. The nozzle area is 0.01 m^2. Assuming the water is directed normal to the plate; determine the horizontal force on the support.



FIND: Horizontal force K_x = ?

Since the force interaction between the fluid and the solid object is the point of interest, we have to use momentum equation.

We must choose a suitable control volume. A number of possible choices are,



Regardless of our choice of control volume, the result should be the same.

I. Use C∀$_I$

Momentum equation in x-direction

$$\vec{F}_x = \vec{F}_{S_x} + \vec{F}_{B_x} = \underbrace{\frac{\partial}{\partial t}\int_{C\forall} u\rho d\forall}_{\substack{=0 \\ Steady\ flow}} + \int_{CS} u\rho\vec{V}\cdot d\vec{A}$$

$\vec{F}_{B_x} = 0$ No body force in x-direction

$$\vec{F}_{S_x} = p_A A - p_A A + R_x$$

$$\vec{F}_{S_x} = \underbrace{p_A A}_{\substack{pressure\,force\\ on\,left\,face}} - \underbrace{p_A A}_{\substack{pressure\,force\\ on\,right\,face}} + \underbrace{R_x}_{\substack{force\,on\,the\,support\\ on\,CV\,(assumed\,positive)}}$$

NOTE: Left and right faces of the control volume are equal.

$$\vec{F}_{S_x} = R_x$$

and

$$R_x = \int_{CS} u\rho\vec{V}\cdot d\vec{A} = \int_{A_1} u\rho\vec{V}\cdot d\vec{A}$$ [No mass crossing top and bottom surfaces, u=0]

$$R_x = \int_{A_1} u\{-|\rho V_1 dA|\}$$ {at 1 $\rho\vec{V}\cdot d\vec{A} = -|\rho V_1 dA|$, since direction of $\vec{V}_1$ and $d\vec{A}_1$ are 180° apart.}

$$R_x = -u_1|\rho V_1 A_1|$$ {properties uniform over A_1}

$$R_x = -15\left[\frac{m}{\sec}\right]\left|999\left[\frac{kg}{m^3}\right]\times 15\left[\frac{m}{\sec}\right]\times 0.01[m]\right|$$

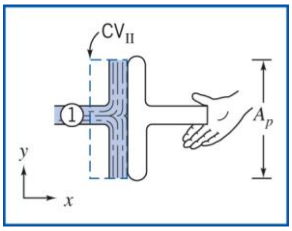
$$R_x = -2.25[kN]$$ {R_x acts opposite to positive direction}

Force on the support $K_x = -R_x = 2.25[kN]$

II. Use C∀ II

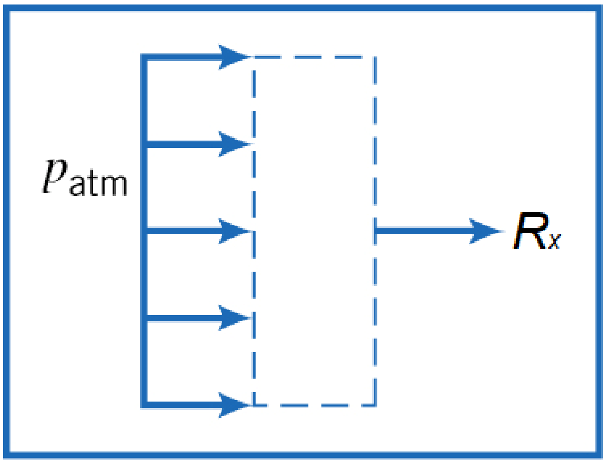
Left and right face areas of the control volume are equal and hence this leads to the equation directly for C∀ I

III. Use C∀ II



Left and right face areas of the control volume are equal.

$$\vec{F}_{S_x} = \int_{CS} u\rho\vec{V}\cdot d\vec{A}$$



for CV_{II}

$$\vec{F}_{S_x} = p_A A + R_x = \int_{A_1} u\rho\vec{V}\cdot d\vec{A} = \int_{A_1} u\left\{-\left|\rho V_1 dA\right|\right\} = -2.25[kN]$$

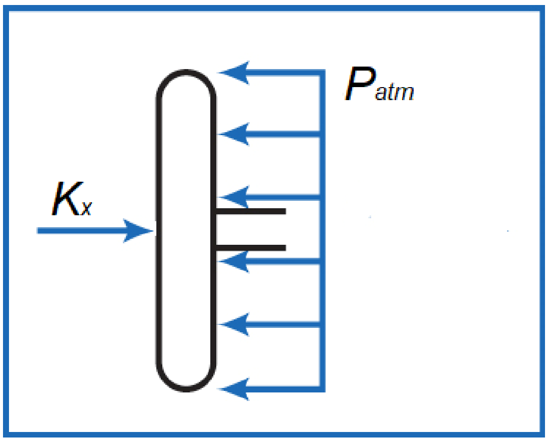
$$p_A A + R_x = -2.25[kN]$$

$$R_x = -p_A A - 2.25[kN]$$

and

$$K_x = -R_x = p_A A + 2.25[kN]$$

To determine the net force on the plate, we need take into account pressure (atmospheric) force of the right face of the plate.

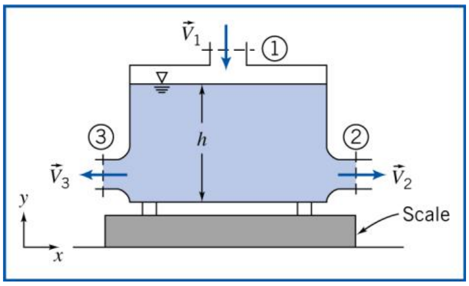


$$F_{net} = K_x - p_A A$$

$$F_{net} = p_A A + 2.25 - p_A A$$

$$F_{net} = 2.25[kN]$$

Example: A metal container, which has a height of 0.6m and an inside cross-sectional area of 0.1 m^2, is placed on a scale. Water flows into the tank at a velocity of 6 m/s through an opening at the top with a cross-sectional area of 0.01 m^2, flows out the openings on the side walls with equal cross-sectional areas. Under steady flow conditions, the height of the water in the tank is 0.5 m. The pressure is atmospheric across all openings, and the container weighs 50 N when it is empty. If the frictional effects are negligible then determine the reading on the scale.



$A_T = 0.1$ m^2

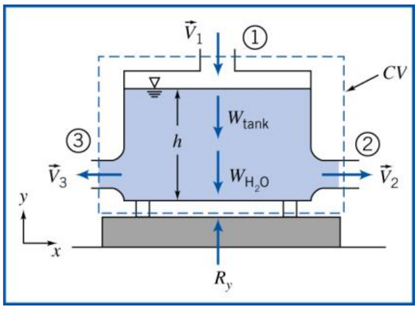
$V_1 = 6$ m/s

$A_1 = 0.01$ m^2

$A_2 = A_3$

h = 0.5 m

$K_y = ?$



The force exerting on the control volume in the y-direction may be found by applying the momentum equation in y-direction.

$$F_{S_y} + F_{B_y} = \underbrace{\frac{\partial}{\partial t}\int_{C\forall} v\rho d\forall}_{\substack{=0 \\ Steady\ flow}} + \int_{CS} v\rho\vec{V}\cdot d\vec{A}$$

$$F_{S_y} + F_{B_y} = \int_{A_1} v\rho\vec{V}\cdot d\vec{A} + \int_{A_2} \underbrace{v}_{=0}\rho\vec{V}\cdot d\vec{A} + \int_{A_3} \underbrace{v}_{=0}\rho\vec{V}\cdot d\vec{A} \qquad (1)$$

$$F_{B_y} = -W_w - W_t = -\rho g h A - W_t$$

W_w : weight of water (2)

W_t : weight of the tank

$$F_{S_y} = R_y - p_{atm}A \qquad (3)$$

Substituting (2) and (3) into (1)

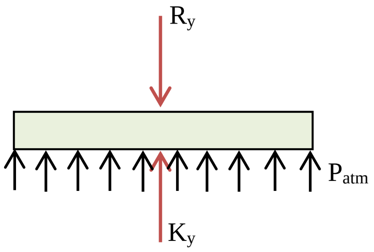
$$R_y - p_{atm}A - W_t - \rho g h A = v_1 \{-|\rho V_1 A_1|\}$$

v_1 = -6 m/s

$\therefore$

$R_y = v_1 \{-|\rho V_1 A_1|\} + p_{atm}A + W_t + \rho g h A$ force exerted by scale on the control volume.

To find the net force acting on scale, consider the free body diagram of the scale



$$K_y = -R_y + p_{atm}A$$

$$K_y = -[v_1\{-|\rho v_1 A_1|\} + p_{atm}A + W_t + \rho g h A] + p_{atm}A$$

$$K_y = -[(-6)\{-|1000 \times 6 \times 0.01|\} + 50 + 1000 \times 9.81 \times 0.5 \times 0.1]$$

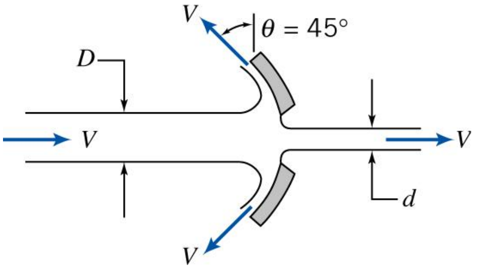
$$K_y = -900.5\ N$$

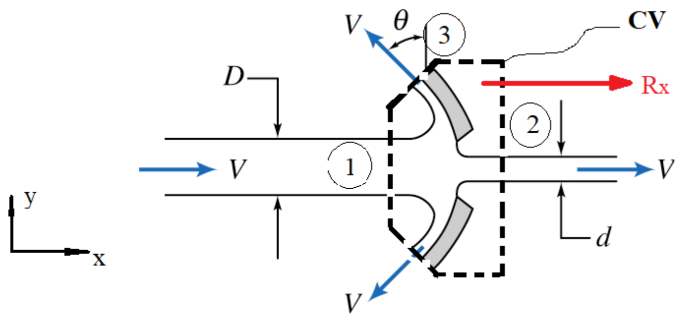
<u>Note</u>: If no water was flowing in, the reading of the scale would be,

$$K_y = -[50\,N + 1000 \times 9.81 \times 0.5 \times 0.1]$$

$$K_y = -540.5\ N$$

Example: A shallow circular dish has a sharp-edged orifice at its center. A water jet of speed **V** strikes the dish concentrically. If the jet issuing from the orifice and from the surface of the dish also has speed **V**, **evaluate the external force needed to hold the dish in place for V = 5 m/s, D=100 mm and d=20 mm.**





Assumptions:
- No body force in x-direction
- Steady flow
- Uniform flow in all sections
- No pressure force
- $V_1 = V_2 = V_3$

Momentum equation in x-direction.

$$F_{S_x} + \underbrace{F_{B_x}}_{\substack{=0 \\ Body\ force}} = \underbrace{\frac{\partial}{\partial t}\int_{C\forall} u\rho d\forall}_{\substack{=0 \\ Steady\ flow}} + \int_{CS} u\rho\vec{V}\cdot d\vec{A}$$

$$R_x = u_1\left\{-\left|\rho V_1 A_1\right|\right\} + u_2\left\{\left|\rho V_2 A_2\right|\right\} + u_3\left\{\left|\rho V_3 A_3\right|\right\}$$

where $u_1 = V,\ u_2 = V,\ u_3 = -V\sin\theta,\ \ A_1 = \dfrac{\pi D^2}{4},\ \ A_2 = \dfrac{\pi d^2}{4},\ \ A_3 = A_1 - A_2$

$$R_x = -\rho V^2\frac{\pi D^2}{4} + \rho V^2\frac{\pi d^2}{4} - \rho V^2(\sin\theta)\frac{\pi}{4}\left(D^2 - d^2\right)$$

$$R_x = \rho V^2\frac{\pi}{4}\left[-D^2 + d^2 - \sin\theta\left(D^2 - d^2\right)\right]$$

$$R_x = \frac{\pi}{4}\rho V^2(1+\sin\theta)\left(d^2 - D^2\right)$$

$$R_x = \frac{\pi}{4}(999)\left(5^2\right)(1+\sin 45)\left(0.02^2 - 0.1^2\right)$$

$R_x = -321.45\ N$ $\quad\Leftarrow$ Force exerted by dish on CV

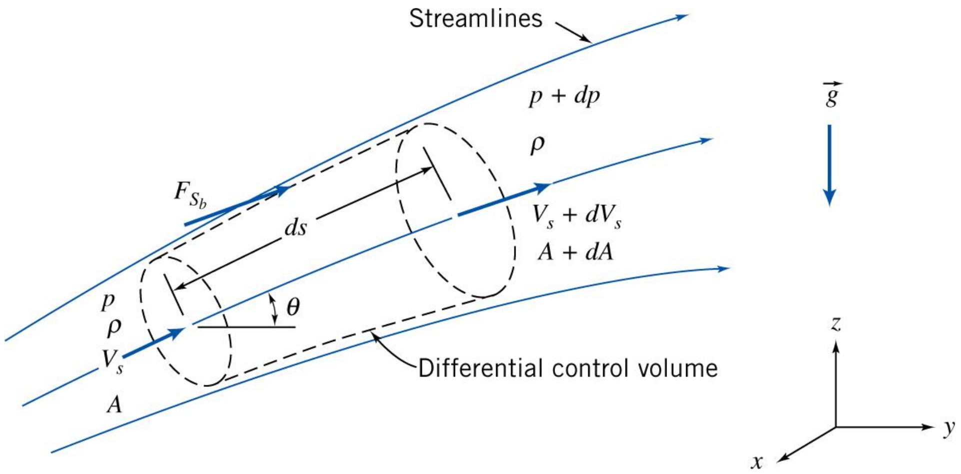
Force acting on dish $K_x = -R_x = 321.45\ N$

Force to hold the dish in place $= -K_x = -321.45\ N$

BERNOULLI'S EQUATION

Bernoulli's equation may be developed as a special form of the momentum or energy equation.

Here, we will develop it as special case of momentum equation. Consider a steady incompressible flow without friction. Apply the control volume equation to the control volume shown.



The control volume chosen is fixed in space and bounded by flow streamlines, and it is thus an element of a stream tube. The length of the control volume is ds.

Because the control volume is bounded by streamlines, the flow across bounding surfaces occurs at the end sections.

The properties at outlet section are assumed to increase by a differential amount.

Continuity Equation

$$\frac{\partial}{\partial t}\int_{C\forall}\rho d\forall + \int_{CS}\rho\vec{V}\cdot d\vec{A} = 0$$

$$\frac{\partial}{\partial t}\int_{C\forall}\rho d\forall = 0[Steady\ flow]$$

$$\{-|\rho V_s A|\} + \{|\rho(V_s + dV_s)(A + dA)|\} = 0$$

or

$$\rho V_s A = \rho(V_s + dV_s)(A + dA) \qquad \text{.......... (1)}$$

<u>s-component of the momentum equation</u>

$$F_{S_s} + F_{B_s} = \frac{\partial}{\partial t}\int_{C\forall} u_s \rho d\forall + \int_{CS} u_s \rho \vec{V} \cdot d\vec{A} \qquad \text{.......... (2)}$$

$$F_{S_s} = pA - (p + dp)(A + dA) + \underbrace{\left(p + \frac{dp}{2}\right)(dA)}_{\substack{\text{pressure force acting} \\ \text{on the bounding stream} \\ \text{surface of the control} \\ \text{volume}}}$$

<u>Note:</u> No friction flow, $R_s = 0$

Then,

$$F_{S_s} = -Adp - \frac{1}{2}dpdA \qquad \text{.......... (3)}$$

The body force component in s-direction is,

$$F_{B_s} = -\rho g_s d\forall = \rho(-g\sin\theta)\left(A + \frac{dA}{2}\right)ds$$

<u>Note:</u>

$$\sin\theta\, ds = dz$$

Therefore,

$$F_{B_s} = -\rho g\left(A + \frac{dA}{2}\right)dz \qquad \text{.......... (4)}$$

The momentum flux will be

$$\int_{CS} u_s \rho \vec{V} \cdot d\vec{A} = V_s\{-|\rho V_s A|\} + (V_s + dV_s)\underbrace{\{|\rho(V_s + dV_s)(A + dA)|\}}_{\textit{from continuity}}$$

From continuity,

$$\left|\rho V_s A\right| = \rho(V_s + dV_s)(A + dA)$$

Hence

$$\int_{CS} u_s \rho \vec{V} \cdot d\vec{A} = V_s(-\rho V_s A) + (V_s + dV_s)(\rho V_s A) = \rho V_s A dV_s \quad \text{...} (5)$$

Substituing Eq. (3), (4), and (5) into (2)

$$-Adp - \underbrace{\frac{1}{2}dpdA}_{\approx 0} - \rho g A dz - \underbrace{\frac{1}{2}\rho g dA dz}_{\approx 0} = \rho V_s A dV_s$$

Dividing by ρA and noting that products of differentials are negligible compared to the remaining terms, we obtain

$$-\frac{dp}{\rho} - gdz = V_s dV_s$$

or

$$-\frac{dp}{\rho} - gdz = d\left(\frac{V_s^2}{2}\right)$$

or

$$\frac{dp}{\rho} + d\left(\frac{V_s^2}{2}\right) + gdz = 0$$

For incompressible flow (ρ = constant), this equation can be integrated to obtain

$$\frac{p}{\rho} + \frac{V_s^2}{2} + gz = \text{constant}$$

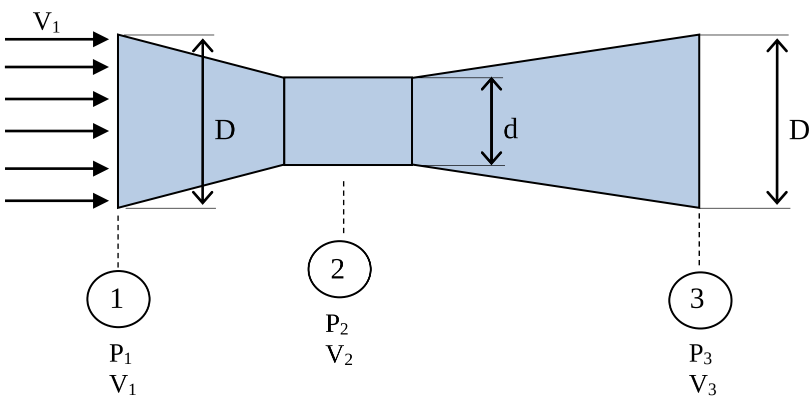
Dropping the subscript *s*,

$$\frac{p}{\rho}+\frac{V^2}{2}+gz=\text{constant}$$ **BERNOULLI EQUATION**

This equation subject to restrictions:

1. Steady flow
2. No friction
3. Flow along a streamline
4. Incompressible flow

Example:



Water at 10°C enters the horizontal venturi tube, shown in the figure, with a uniform and steady velocity of 2.0 m/s and an inlet pressure of 150 kPa. Find the pressure at the throat, 2, where d = 3.0 cm and at the exit where D = 6.0 cm.

Find: P_2 = ? and P_3 = ?

Assumptions:

- Incompressible flow
- Negligible friction
- Steady flow

Solution:

We assume constant density and uniform velocity over planes 1 and 2.

Applying the continuity equation between plane 1 and 2, we obtain

$$V_1 A_1 = V_2 A_2 \qquad \Rightarrow V_2 = \left(\frac{A_1}{A_2}\right) V_1 = \left(\frac{D}{d}\right)^2 V_1$$

Applying Bernoulli equation to a streamline connecting cross-sections 1 and 2,

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + g z_2$$

Assuming that $z_1 = z_2$ and solving for p_2, we obtain

$$p_2 = p_1 + \frac{\rho}{2}\left(V_1^2 - V_2^2\right)$$

Substituting $V_2 = \left(\frac{D}{d}\right)^2 V_1$

$$p_2 = p_1 + \frac{\rho}{2}\left[1 - \left(\frac{D}{d}\right)^4\right] V_1^2$$

$$p_2 = 150 \times 10^3 \left[\frac{N}{m^2}\right] + \frac{1000\left[\frac{kg}{m^3}\right]}{2}\left[1 - \left(\frac{6[cm]}{3[cm]}\right)^4\right]\left[2\left[\frac{m}{s}\right]\right]^2$$

$$p_2 = 150 \times 10^3 \left[\frac{N}{m^2}\right] - 30 \times 10^3 \left[\frac{N}{m^2}\right]$$

$$p_2 = 120 \times 10^3 \left[\frac{N}{m^2}\right] = 120\left[kPa\right]$$

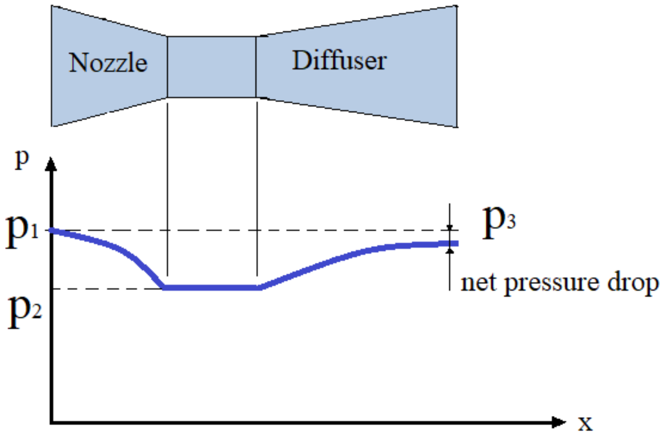
Similarly, applying the continuity equation and Bernoulli equation between planes 1 and 3, we can obtain p_3.

$$\left.\begin{aligned} V_1 A_1 &= V_3 A_3 \\ A_1 &= A_3 \end{aligned}\right\} \quad \Rightarrow V_1 = V_3$$

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_3}{\rho} + \frac{V_3^2}{2} + gz_3$$

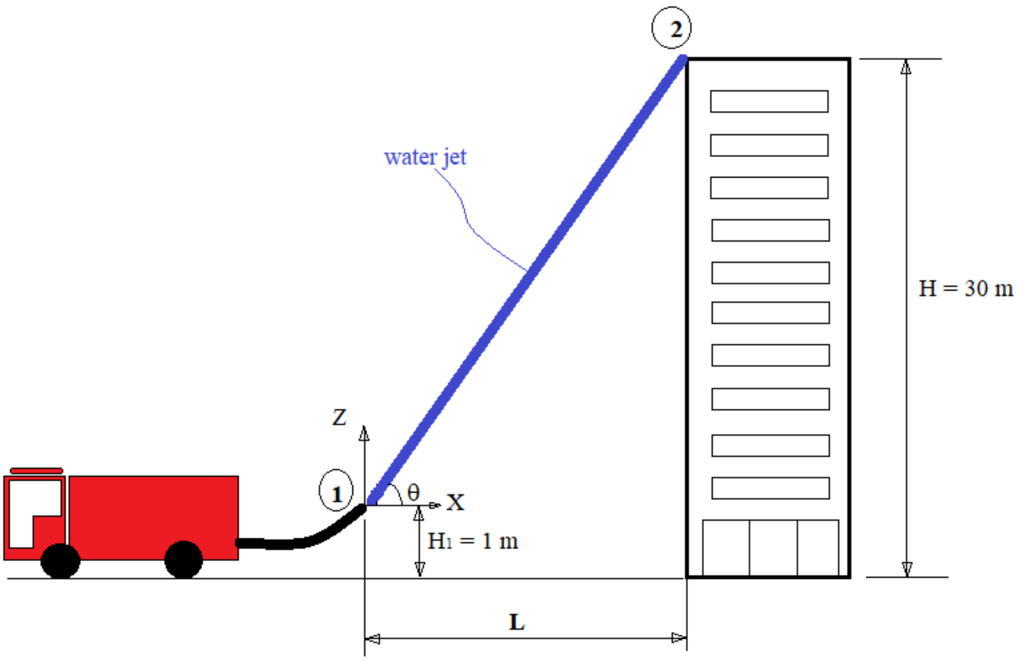
$$z_1 \cong z_3$$

$$\frac{p_1}{\rho} = \frac{p_3}{\rho} \quad \Rightarrow p_3 = p_1 = 150\left[kPa\right]$$



Because, we assumed the water is inviscid, the static pressure drop (p_1-p_2) is fully recovered in the diffuser by decreasing the fluid velocity to V_1. However, full pressure recovery would not occur in a real venturi tube. Viscous effects would produce a net pressure drop between **1** and **3**.

Example: A city has a fire truck whose pump and hose can deliver **60 lt/sec** with nozzle velocity of **36 m/sec**. The tallest building in the city is **30 m** high. The firefighters hold the nozzle at an angle of **75°** from the ground. Find the minimum distance the firefighters must stand from the building to put out a fire on the roof without the aid of a ladder. The firefighters hold the hose **1 m** above the ground. Assume that the water velocity is not reduced by air resistance.



Given:
Q = 60 lt/s
V_1 = 36 m/s
H = 30 m
$\theta = 75°$
H_1 = 1 m

Find:
L = ?

Assumptions:
- Steady flow
- There is no friction
- Incompressible flow

The slope of the water jet is $\frac{dx}{dz} = \frac{V_x}{V_z}$

[NOTE: The centerline of the water jet is a streakline, pathline and a streamline]

Writing Bernoulli equation between points **1** and **any point** on jet

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p}{\rho} + \frac{V^2}{2} + gz$$

NOTE: at any point $p = p_1 = p_{atm}$
Taking $z_1 = 0$ and solving for V.

$$V^2 = V_1^2 - 2gz \,, \; z_2 = 30 - 1 = 29[m]$$

With negligible air resistance, there is no force on the fluid in x-direction. Hence,

$$V_x = V_{1x} = V_1 \cos\theta$$

$$\therefore \quad V^2 = V_x^2 + V_z^2 = V_1^2 - 2gz$$

$$V_1^2 \cos^2\theta + V_z^2 = V_1^2 - 2gz$$

$$V_z^2 = V_1^2 \underbrace{\left(1 - \cos^2\theta\right)}_{\sin^2\theta} - 2gz$$

$$V_z^2 = \sqrt{V_1^2 \sin^2\theta - 2gz}$$

$\therefore$ The jet trajectory equation

$$\frac{dx}{dz} = \frac{V_x}{V_z} = \frac{V_1 \; \cos\theta}{\left(V_1^2 \sin^2\theta - 2gz\right)^{1/2}}$$

multiplying by dz and integrating gives

$$x = V_1 \cos\theta \frac{V_1 \cos\theta - \sqrt{V_1^2 \sin^2\theta - 2gz}}{g}$$

rearrangement gives,

$$x = \frac{V_1^2 \sin 2\theta}{2g}\left(1 - \sqrt{1 - \frac{2gz}{V_1^2 \sin^2\theta}}\right)$$

substituting, z = 29 m, θ = 75°, V_1 = 36 m/s

$$x = \frac{36^2 \sin 150}{2(9.81)}\left(1 - \sqrt{1 - \frac{2(9.81)(29)}{36^2 \sin^2 75}}\right)$$

$$x \cong 9\ [m]$$

MOMENT OF MOMENTUM
(The Angular Momentum Equation)

To derive the moment of momentum equation we use the similar method that we use for derivation of continuity and momentum equation, i.e., first we write moment of momentum for a system, then obtain an equation for the control volume.

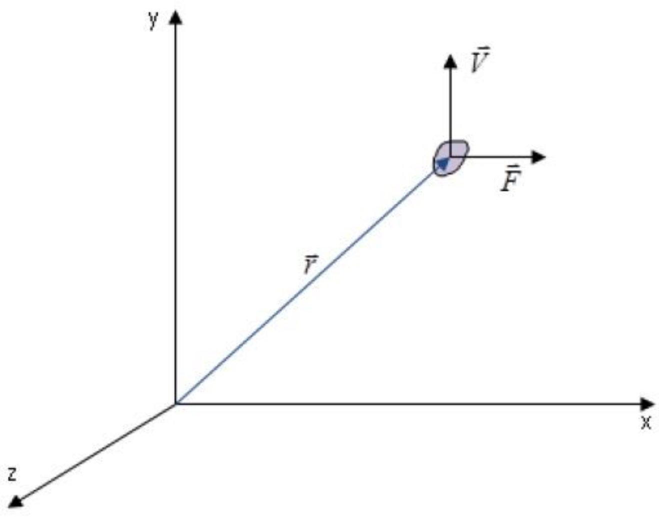
Moment of momentum for a system is

$$\vec{T} = \left.\frac{d\vec{H}}{dt}\right)_{sys} \tag{1}$$

where $\vec{T}$: Total torque exerted on the system by its surrounding
$\vec{H}$: Angular momentum of the system

$$\vec{H} = \int_{M(sys)} \vec{r} \times \vec{V} dm = \int_{\forall(sys)} \vec{r} \times \vec{V} \rho d\forall \tag{2}$$

The position vector $\vec{r}$, locates each mass and or volume element of the system with respect to the coordinate system.



The torque $\vec{T}$ applied to a system may be written

$$\vec{T} = \underset{\substack{\text{Torque due to}\\ \text{surface forces}}}{\vec{r} \times \vec{F}_s} + \underbrace{\int_{M\,sys} \vec{r} \times g dm}_{\substack{\text{Torque due to}\\ \text{body forces}}} + \underset{\substack{\text{Torque applied}\\ \text{by a shaft}}}{\vec{T}_{shaft}} \tag{3}$$

The relation between the system and fixed control volume formulation is

$$\left.\frac{dN}{dt}\right)_{sys} = \frac{\partial}{\partial t}\int_{C\forall}\eta\rho d\forall + \int_{CS}\eta\rho\vec{V}\cdot d\vec{A}$$
$$N_{sys} = \int_{M\,sys}\eta dm = \int_{\forall\,sys}\eta d\forall \qquad (4)$$

and setting $N = \vec{H}$ and $\eta = \vec{r}\times\vec{V}$, then

$$\left.\frac{d\vec{H}}{dt}\right)_{sys} = \frac{\partial}{\partial t}\int_{C\forall}\vec{r}\times\vec{V}\rho d\forall + \int_{CS}\vec{r}\times\vec{V}\rho\vec{V}\cdot d\vec{A} \qquad (5)$$

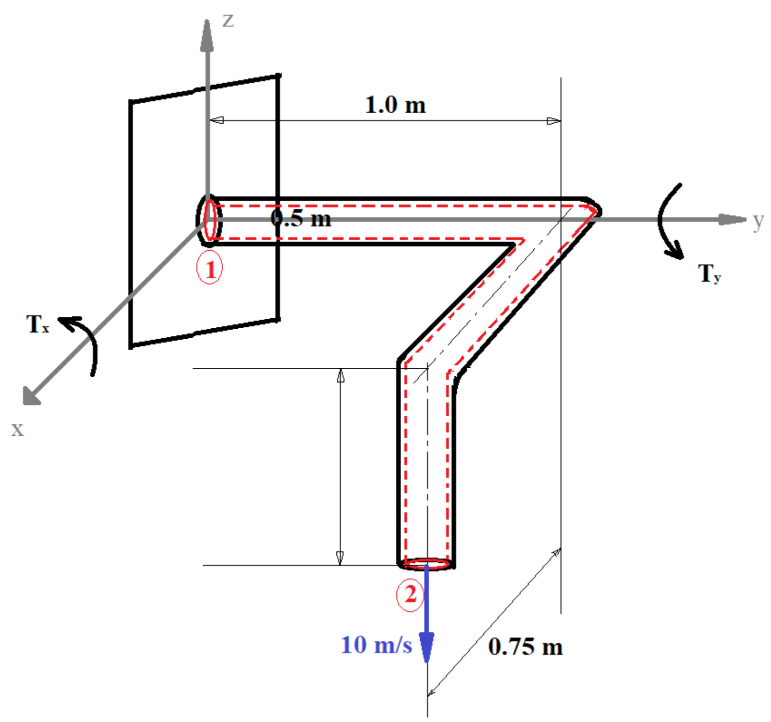
Combining Eqs. (1), (3), and (5), we obtain

$$\underbrace{\vec{r}\times\vec{F}_s + \int_{M\,sys}\vec{r}\times g\rho d\forall + \vec{T}_{shaft}}_{\text{Torque acting on control volume}} = \underbrace{\frac{\partial}{\partial t}\int_{C\forall}\vec{r}\times\vec{V}\rho d\forall + \int_{CS}\vec{r}\times\vec{V}\rho\vec{V}\cdot d\vec{A}}_{\text{Rate of change of angular momentum}}$$

$$\vec{r}\times\vec{F}_s + \int_{M\,sys}\vec{r}\times g\rho d\forall + \vec{T}_{shaft} = \frac{\partial}{\partial t}\int_{C\forall}\vec{r}\times\vec{V}\rho d\forall + \int_{CS}\vec{r}\times\vec{V}\rho\vec{V}\cdot d\vec{A}$$

Moment of momentum equation for an inertial control volume

Example: Consider the pipe mounted on a wall shown in figure. The pipe inside diameter is **20 cm**, and both pipe bends are **90°**. Water enters the pipe at the base and exits at the open end with a speed of **10 m/s**. **Calculate the torsional moment and the bending moment at the base of the pipe**. Neglect the weight of water and pipe.



Assumptions:
- Incompresible flow
- Flow is uniform at all cross-sections.
- Steady flow
- Negligible body force

Find:
- Torsional moment T_y =?
- Bending moment T_x =?

Writing the moment of momentum equation

$$\vec{r}\times\vec{F}_s + \underbrace{\int_{M\,sys} \vec{r}\times g\rho d\forall}_{\substack{=0\\ negligible}} + \vec{T}_{shaft} = \underbrace{\frac{\partial}{\partial t}\int_{C\forall} \vec{r}\times\vec{V}\rho d\forall}_{\substack{=0\\ steady}} + \int_{CS} \vec{r}\times\vec{V}\rho\vec{V}\cdot d\vec{A}$$

$$\vec{r}_1\times\left(\vec{R}_{x1} + p_1 A_1\vec{i}\right) + \vec{r}_2\times\left(p_2 A_2\vec{k}\right) + T_x\vec{i} + T_y\vec{j} = \int_{A_1} \vec{r}_1\times\vec{V}\rho\vec{V}\cdot d\vec{A} + \int_{A_2} \vec{r}_1\times\vec{V}\rho\vec{V}\cdot d\vec{A}$$

$$\vec{r}_1 = 0, \qquad \vec{r}_2 = 0.75\vec{i} + 1\vec{j} - 0.5\vec{k}\ [m]$$

$$\vec{V}_1 = 10\vec{j}\ [m/s],\ \ \vec{V}_2 = -10\vec{k}\ [m/s],\ \ p_{2gage} = 0$$

$$T_x\vec{i} + T_y\vec{j} = \left(0.75\vec{i} + 1\vec{j} - 0.5\vec{k}\right)\times\left(-10\vec{k}\right)(1000)\left|10\frac{\pi D^2}{4}\right|$$

$$T_x\vec{i} + T_y\vec{j} = 2356\vec{j} - 3142\vec{i}\ [Nm]$$

$\therefore$

$$\left.\begin{array}{ll} \text{Bending moment} & T_x = -3142\ [Nm] \\ \text{Torsional moment} & T_y = 2356\ [Nm] \end{array}\right\}\text{Moment applied to the control volume by the base}$$

Moment acting on the base

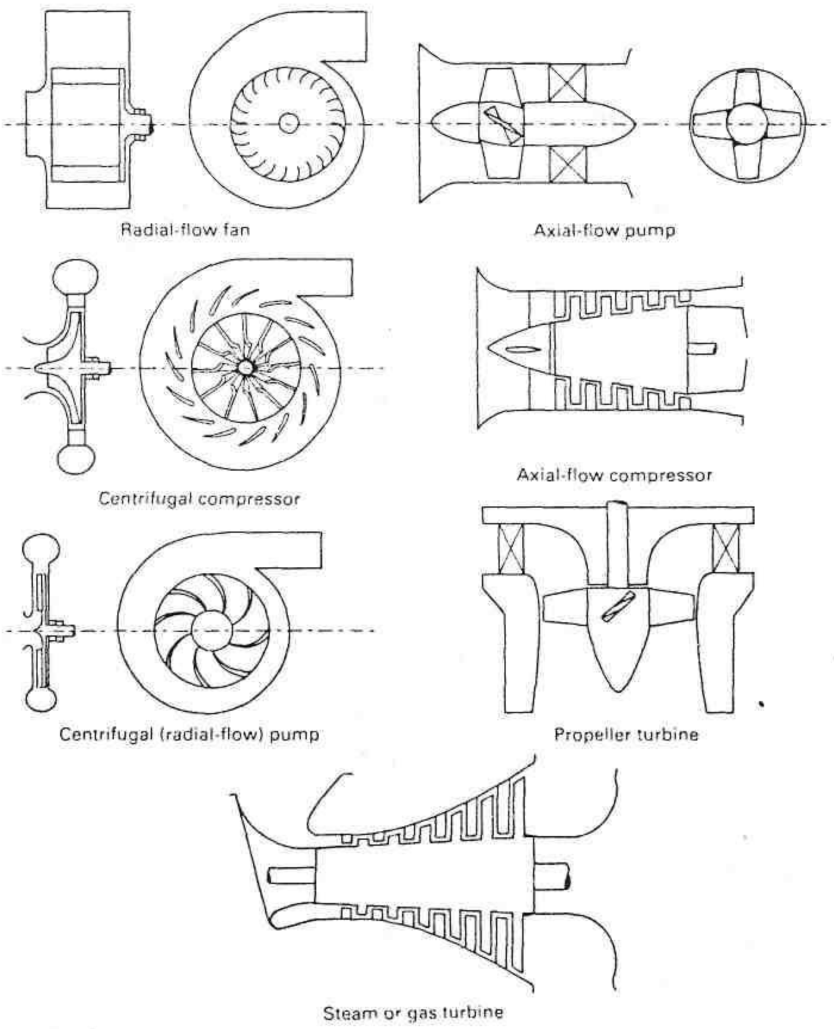
$$\left.T_x\right)_B = -T_x = 3142\ [Nm]$$

$$\left.T_y\right)_B = -T_y = -2356\ [Nm]$$

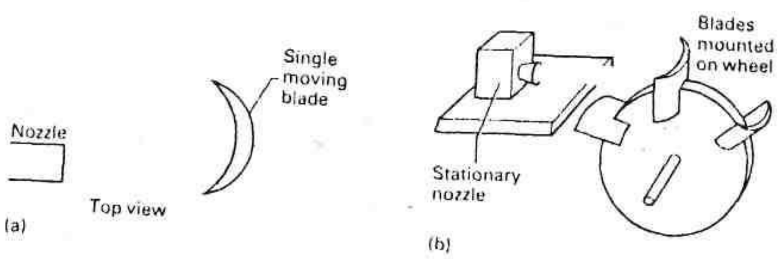
APPLICATION TO TURBOMACHINERY

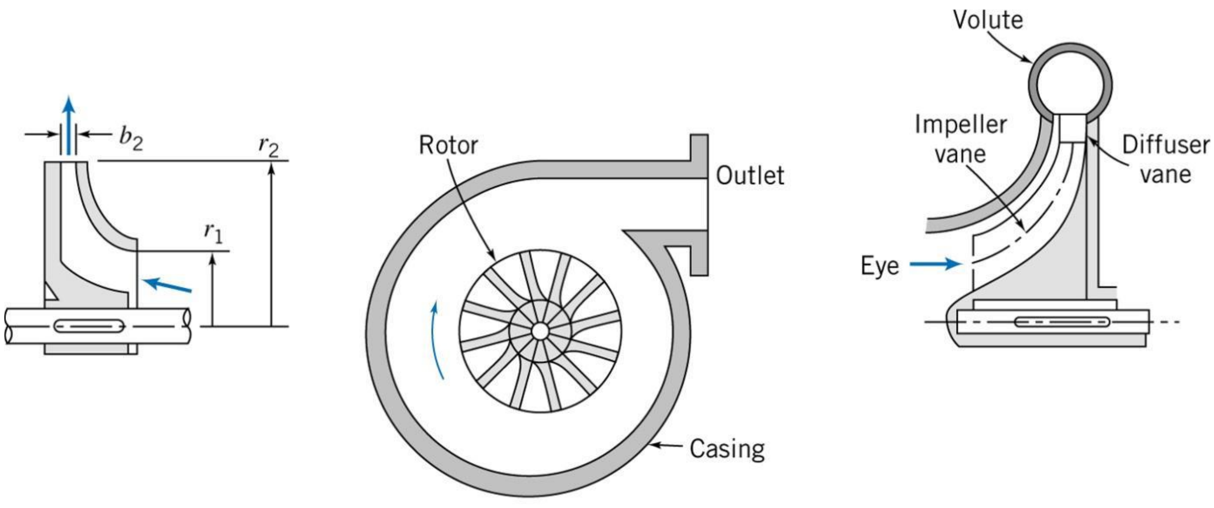
The equation of moment of momentum is used for analysis of rotating machinery. **A turbomachine is a device that uses a moving rotor, carrying a set of blades or vanes, to transfer work to or from a moving stream of fluid**. If the work is done on the fluid by the rotor, the machine is called a **pump** or **compressor**. If the fluid delivers work to rotor, the machine is called a **turbine**.

Figure Sketches of typical turbomachines.

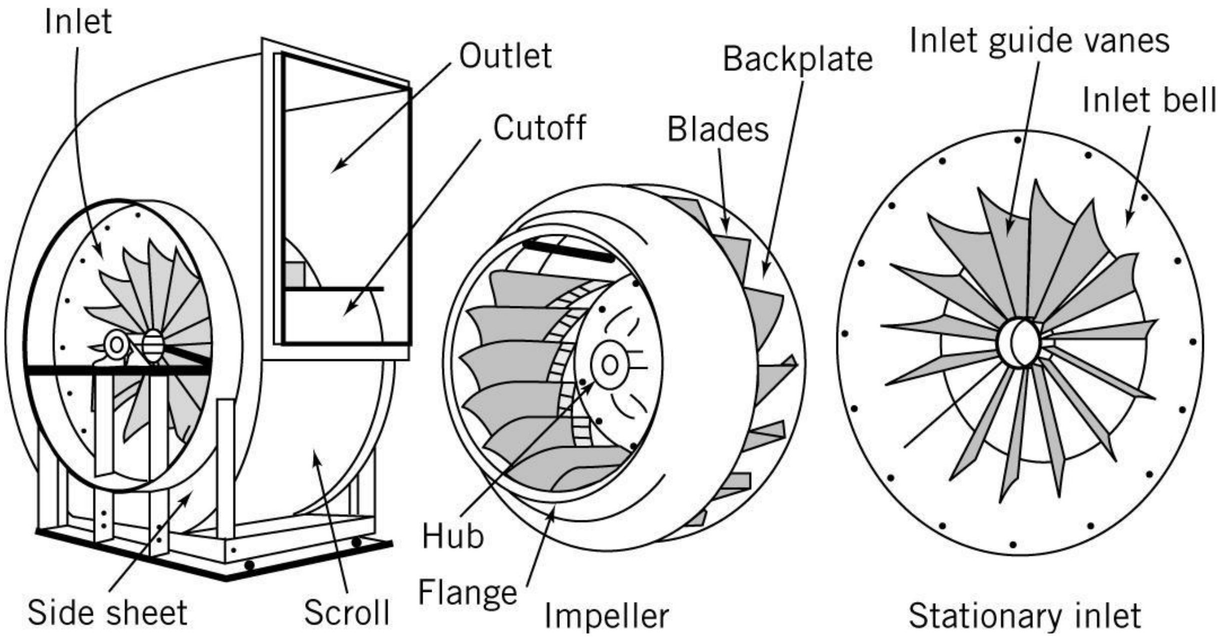


A simple turbine made of several moving vanes for fluid jet deflection.



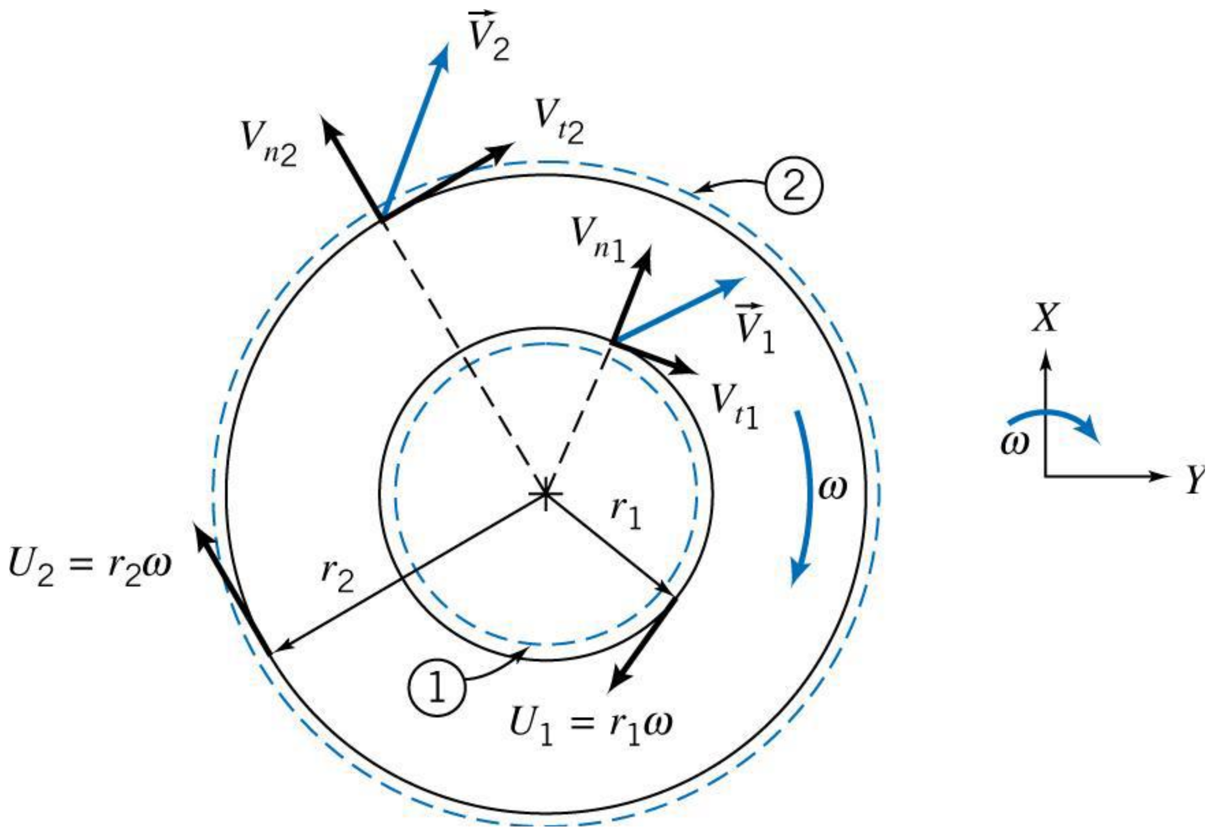


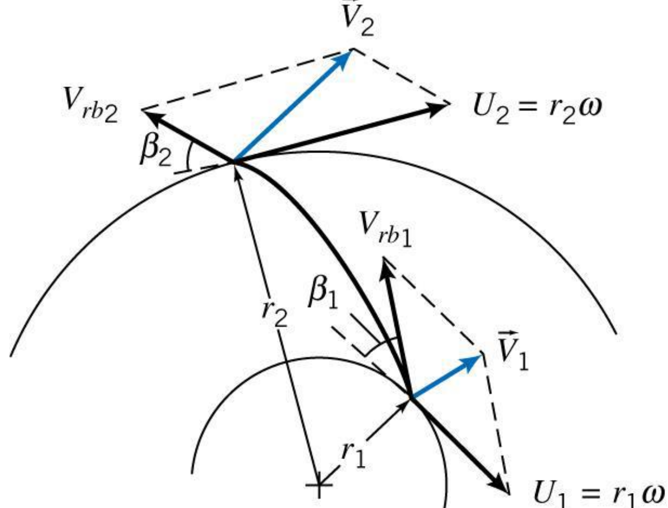
(a) Centrifugal pump (b) Centrifugal blower (c) Centrifugal compressor



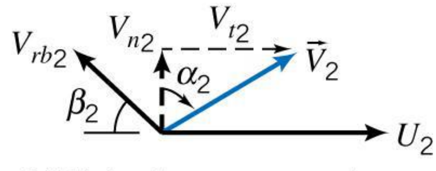
Turbomachines are classified as **axial flow, radial flow or mixed flow** depending on the direction of fluid motion with respect to the rotor's axis of rotation as the fluid passes over the blades. In an axial-flow rotor, the fluid maintains an essentially constant radial position as it flows from rotor inlet and to rotor outlet. In a radial-flow rotor, the fluid moves primarily radially from rotor inlet to rotor outlet although fluid may be moving in the axial direction at the machine inlet or outlet. In the mixed-flow rotor, the fluid has both axial and radial velocity components as it passes through the rotor.

For turbomachinery analysis, it is convenient to choose a fixed control volume enclosing the rotor for analysis of torque reaction.

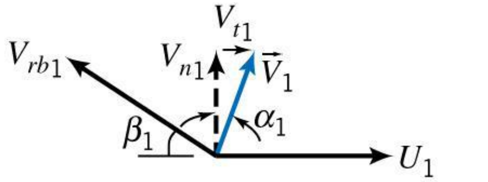




(*a*) Absolute velocity as sum of velocity relative to blade and rotor velocity



(*c*) Velocity components at outlet



(*b*) Velocity components at inlet

The angle of the absolute fluid velocity α is measured from the normal.

Blade angles β are measured relative to the circumferential direction.

As a first approximation, torques due to surface forces may be ignored. The torques due to body forces may be neglected by symmetry. Then for a steady flow, moment of momentum equation becomes

$$\vec{T}_{shaft} = \int_{CS} \vec{r} \times \vec{V} \rho \vec{V} \cdot d\vec{A} = \int_{inlet} \vec{r} \times \vec{V} \rho \vec{V} \cdot d\vec{A} + \int_{outlet} \vec{r} \times \vec{V} \rho \vec{V} \cdot d\vec{A}$$

Taking the coordinate system in such a way that z-axis is aligned with the axis of rotation of the machine, and assuming that at the rotor inlet and outlet flow is uniform, we get

$$\vec{T}_{shaft} = \left(r_2 V_{t2} - r_1 V_{t1} \right) \dot{m} \vec{k}$$

or in scalar form

$$T_{shaft} = \left(r_2 V_{t2} - r_1 V_{t1} \right) \dot{m}$$ **EULER TURBINE RQUATION**

where V_{t1} and V_{t2} are tangential components of the **absolute** fluid velocity crossing the control surface at inlet and outlet, respectively.

The rate of work done on a turbomachinery rotor is

$$\dot{W}_{in} = \vec{\omega} \cdot \vec{T}_{shaft} = \omega \vec{k} \cdot T_{shaft} \vec{k} = \omega T_{shaft}$$
$$\dot{W}_{in} = \omega \left(r_2 V_{t2} - r_1 V_{t1} \right) \dot{m}$$

NOTE: $\omega r = U$ tangential velocity of the rotor.

Dividing both sides by $\dot{m} g$, we obtain **head added** to the flow.

$$\Delta h = \frac{\dot{W}_{in}}{\dot{m}g} = \frac{1}{g}\left(U_2 V_{t2} - U_1 V_{t1}\right) \quad [m]$$

The above equation suggest that fluid velocity at inlet and outlet and also rotor velocity should be defined clearly. It is useful to develop velocity polygons for the inlet and outlet flows.

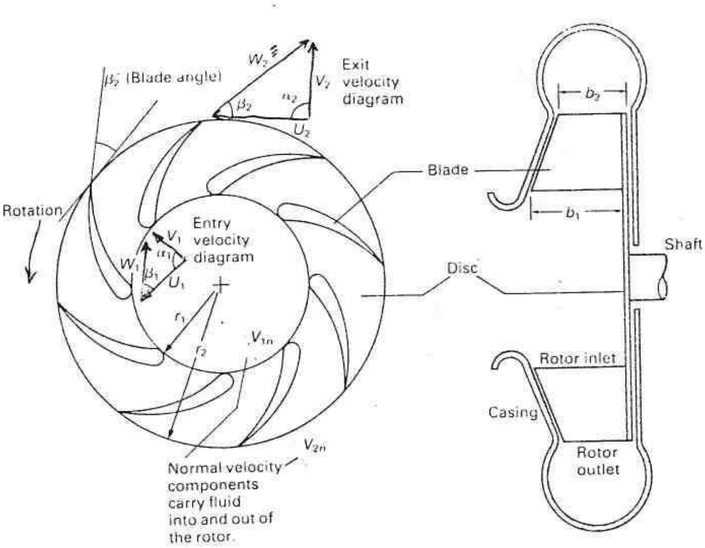
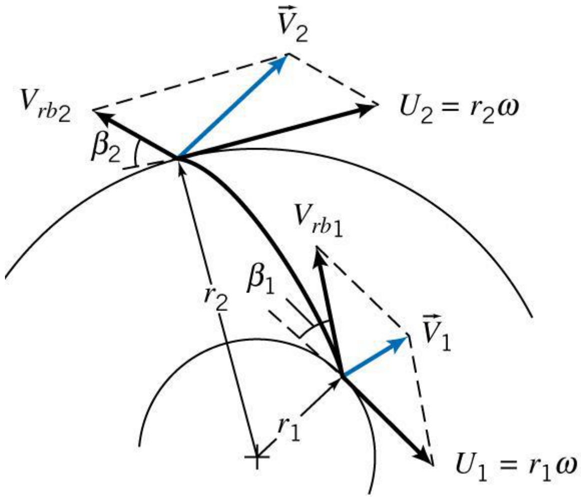
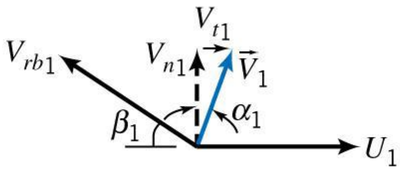


Figure 4.30 Details of flow entering and leaving a radial-flow rotor.



Blade angles β are measured relative to the circumferential direction.

<u>Velocity polygon at inlet</u>



(b) Velocity components at inlet

At the inlet the absolute velocity of the fluid $\vec{V}_1$ is equal to vectoral sum of the fluid velocity with respect to blade and the tangential velocity of the rotor, i.e.

$$\vec{V}_1 = \vec{U}_1 + \vec{V}_{rb1}$$

$\vec{V}_{n1}$ is the normal component of the fluid velocity which is also normal to the flow area.

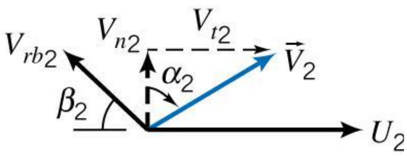
The angle of the absolute fluid velocity α is measured from the normal.

Note: $\vec{V}_{n1} = \vec{V}_{rb1}$

Velocity polygon at inlet

A similar velocity polygon can also be developed for the outlet such that

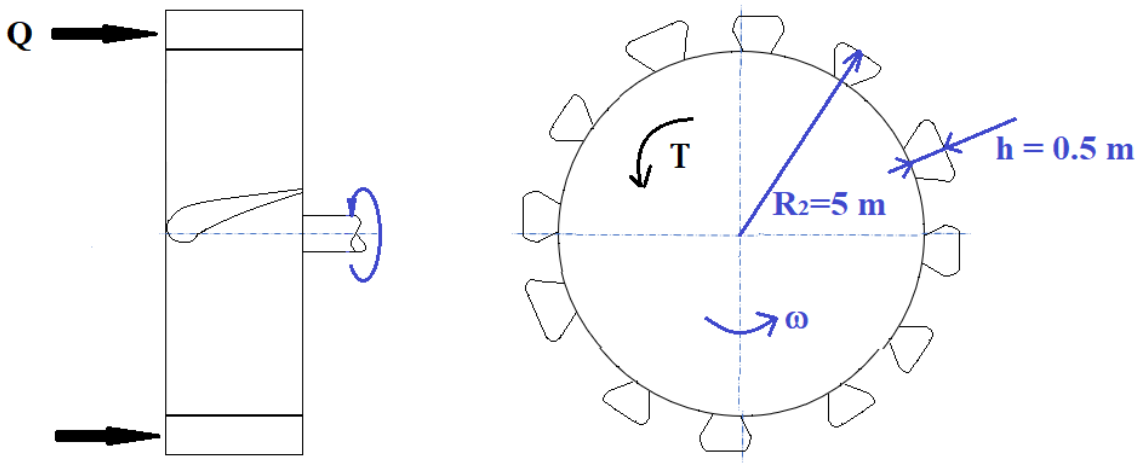
$$\vec{V}_2 = \vec{V}_{rb2} + \vec{U}_2$$

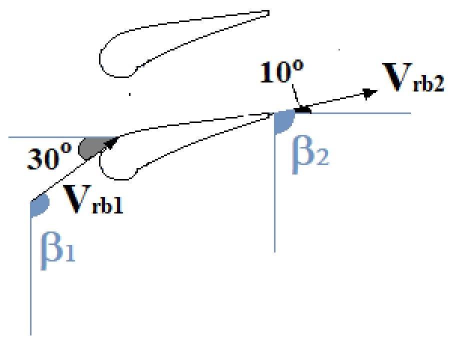


(c) Velocity components at outlet

The inlet and outlet velocity polygons provide all the information required to calculate the torque or power absorbed or delivered by the impeller. The resulting values represent the performance of a turbomachine under **idealized conditions** at the design operating point; since we have assumed that all flows are uniform and that they enter and leave the rotor tangent to blades.

Example: The axial-flow hydraulic turbine has a water flow rate of **75 m^3/s**, an outer radius **R = 5.0 m**, and a blade height **h = 0.5 m**. Assume uniform properties and velocities over both the inlet and the outlet. The water temperature is **20°C**, and the turbine rotates at **60 rpm**. The relative velocities Vr1 and Vr2 make angles of **30°** and **10°**, respectively, with the normal to the flow area. Find the output torque and power developed by the turbine.





<u>**Given:**</u>

Q = 75 m^3/s

R = 5.0 m

h = 0.5 m

water temperature 20°, ρ = 998 kg/m^3

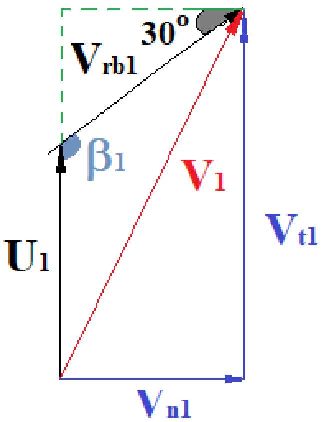
<u>**Find:**</u> T = ?, $\dot{W}$ = ?

$$T = \dot{m}\left(R_2 V_{t2} - R_1 V_{t1}\right)$$
$$\dot{m} = \rho g = (998)(9.81) = 74850\ [kg/s]$$

For an axial flow machine where the blade height h is small compared to the diameter, an average radius may be utilized.

$$R_1 = R_2 \cong R - \frac{1}{2}h = 5 - \frac{1}{2}0.5 = 4.75[m]$$

The tangential components of the absolute velocity can be calculated from the velocity triangles,



$$\vec{V}_1 = \vec{U}_1 + \vec{V}_{rb1}$$

$$U_1 = R_1\omega = 4.75[m]\frac{60(2\pi)}{60}[rad/\sec]$$
$$U_1 = 29.85[m/s]$$

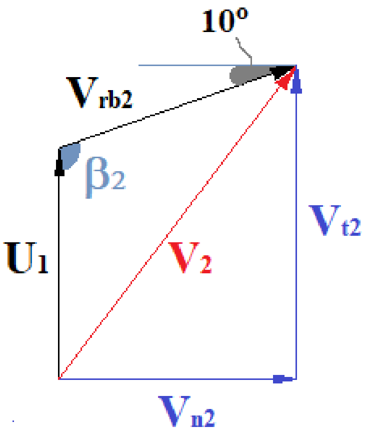
$$V_{n1} = \frac{Q}{A_1} = \frac{Q}{\pi R_1^2 - \pi R_2^2} = \frac{Q}{\pi(R_1^2 - R_2^2)} = \frac{75}{\pi(5.0^2 - 4.5^2)}$$
$$V_{n1} = 5.03[m/s]$$

From the velocity triangle,

$$V_{t1} = U_1 + V_{rb1}\cos\left(180 - \beta_1\right) = U_1 - V_{rb1}\cos\beta_1$$
$$V_{n1} = \left(V_n\right)_{rb1} = V_{rb1}\sin\left(180 - \beta_1\right) = V_{rb1}\sin\beta_1 \quad \Rightarrow \quad V_{rb1} = \frac{V_{n1}}{\sin\beta_1}$$

$$\therefore \qquad V_{t1} = U_1 - V_{n1}\frac{1}{\tan\beta_1}$$
$$V_{t1} = 29.85 - \frac{5.03}{\tan 120} = 32.75[m/s]$$

Similarly, for the outlet



$$\beta_2 = 100°$$
$$V_{n1} = V_{n2} = 5.03[m/s]$$
$$V_{n2} = (V_n)_{rb2}$$
$$U_2 = U_1 = 29.85\,[m/s]$$

$$V_{t2} = U_2 + V_{rb2}\cos(\pi - \beta_2) = U_2 - V_{rb2}\cos\beta_2$$
$$V_{n2} = (V_n)_{rb2} = V_{rb2}\sin(\pi - \beta_1) = V_{rb2}\sin\beta_2 \quad \Rightarrow \quad V_{rb2} = \frac{V_{n2}}{\sin\beta_2}$$

$$\therefore \qquad V_{t2} = U_2 - V_{n2}\frac{1}{\tan\beta_2}$$
$$V_{t1} = 29.85 - \frac{5.03}{\tan 100} = 30.74[m/s]$$

Hence,

$$T = \dot{m}\left(R_2 V_{t2} - R_1 V_{t1}\right) = 74850\,[kg/s]\left[4.75[m]30.74[m/s] - 4.75[m]32.75[m/s]\right]$$
$$T = -7.15\times10^5\,[Nm]$$

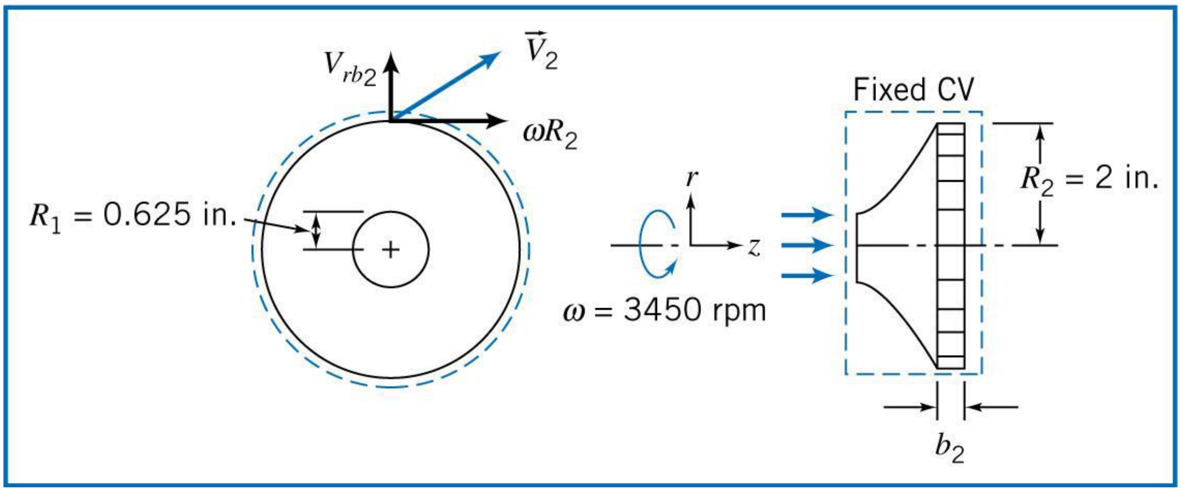
The significance of the negative sign is that the torque is in direction opposite that assumed to be positive. (T is load torque that resists rotation of the turbine.)

The magnitude is $\quad T = 7.15\times10^5\,[Nm]$

The power output

$$\dot{W} = \omega T = \frac{60(2\pi)}{60}(7.15\times10^5)$$
$$\dot{W} = 4.49\times10^6\,[Nm/s]$$
or
$$\dot{W} = 4490[kW]$$

<u>Example:</u> Water at **0.6 m3/min** enters a mixed-flow pump impeller axially through a **5 cm** diameter inlet. The inlet velocity is axial and uniform. The outlet diameter of the impeller is **10 cm**. Flow leaves he impeller at a velocity of **3 m/s** relative to the radial blades. The impeller speed is **3450 rpm**. Determine the impeller exit width **b**, the torque input to the impeller and the horsepower supplied.



<u>Assumptions:</u>

- Neglect torques due to body and surface forces

- Steady flow

- Uniform flow at the inlet and outlet sections

- Incompressible flow

<u>Find:</u>

$b_2 = ?$

$\dot{W}_{in} = ?$

Continuity equation $\underbrace{\frac{\partial}{\partial t}\int_{C\forall} \rho d\forall}_{=0} + \int_{CS} \rho\vec{V}\cdot d\vec{A} = 0$

$$\int_{CS} \rho\vec{V}\cdot d\vec{A} = 0$$

$$\int_{1} \rho\vec{V}\cdot d\vec{A} + \int_{2} \rho\vec{V}\cdot d\vec{A} = 0$$

$$\left\{-\left|\rho V_1 A_1\right|\right\} + \left\{\left|\rho V_2 A_2\right|\right\} = 0$$

$$\left\{-\left|\rho V_1 \pi R_1^2\right|\right\} + \left\{+\left|\rho V_{rb2} 2\pi R_2 b_2\right|\right\} = 0$$

$$\rho V_{rb2} 2\pi R_2 b_2 = \rho Q$$

$$\Rightarrow \qquad b_2 = \frac{Q}{2\pi R_2 V_{rb2}} = \frac{\frac{0.6}{60}\left[\frac{m^3}{s}\right]}{2\pi\left(0.1[m]\right)\left(3\left[\frac{m}{s}\right]\right)} = 0.0106[m]$$

$$b_2 = 0.0106[m]$$

NOTE: V_{rb2} is normal to the flow area.

T_{shaft} = ?

From Euler turbine equation

$$T_{shaft} = \left(R_2 V_{t2} - R_1 V_{t1}\right)\dot{m}$$

$$V_{t1} = 0, \qquad V_{t2} = U_2 = \omega R_2$$

$$\therefore T_{shaft} = R_2 \omega R_2\, \dot{m} = R_2^2 \omega \rho Q = \left(0.05\right)^2 \frac{2\pi(3450)}{60}\left(1000\right)\frac{0.6}{60}$$

$$T_{shaft} = 9.03[Nm]$$

$$\dot{W}_{in} = ?$$

$$\dot{W} = \omega T = \frac{2\pi\left(3450\right)}{60}(9.03)$$

$$\dot{W} = 3262.4[W]$$

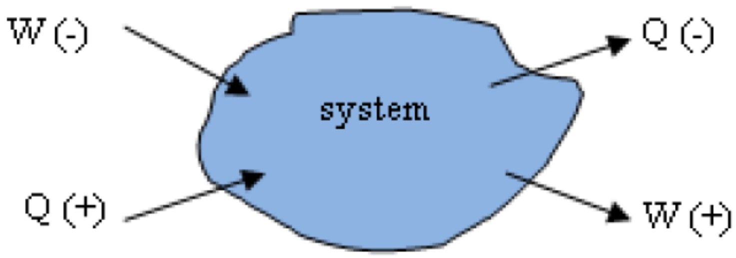
$$\dot{W} = \frac{3262.4}{745.7} = 4.375[HP]$$

THE FIRST LAW OF THERMODYNAMICS

(Energy Equation for a Control Volume)

We obtain the general energy equation by combining the first law of thermodynamics and the transport theorem. **For a system**, conservation of energy can be written as,

$$\begin{bmatrix}\text{Net rate of transfer of} \\ \text{energy to the system}\end{bmatrix} = \begin{bmatrix}\text{Rate of change of} \\ \text{the energy of a system}\end{bmatrix}$$



with mathematical terms

$$\dot{Q} - \dot{W} = \left.\frac{dE}{dt}\right)_{system} \tag{1}$$

<u>NOTE:</u> Heat and work are both energies. In general energy can be classified in two groups

1. Mechanical Energy
   - Work
   - Kinetic energy
   - Potential energy
2. Thermal Energy
   - Heat
   - Internal energy

- Chemical energy
- Nuclear energy

Mechanical energies are associated with force and motion. Thermal energies are associated with temperature, molecular structure and heat transfer.

$$E_{sys} = \int_{\substack{mass \\ (sys)}} edm = \int_{\forall\ sys} e\rho d\forall$$

and

$$e = u + \frac{v^2}{2} + gz \qquad (2)$$

The system and control volume formulations are related by Reynolds transport theorem

$$\left.\frac{dN}{dt}\right)_{sys} = \frac{\partial}{\partial t}\int_{C\forall} \eta\rho d\forall + \int_{CS} \eta\rho\vec{V}\cdot d\vec{A}$$

To derive energy equation for a control volume, we set ***N=E* and *η=e***, then

$$\left.\frac{dE}{dt}\right)_{sys} = \frac{\partial}{\partial t}\int_{C\forall} e\rho d\forall + \int_{CS} e\rho\vec{V}\cdot d\vec{A} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Note that in deriving transport equation, the system and control volume coincided at ***t = t0*, hence** we can write

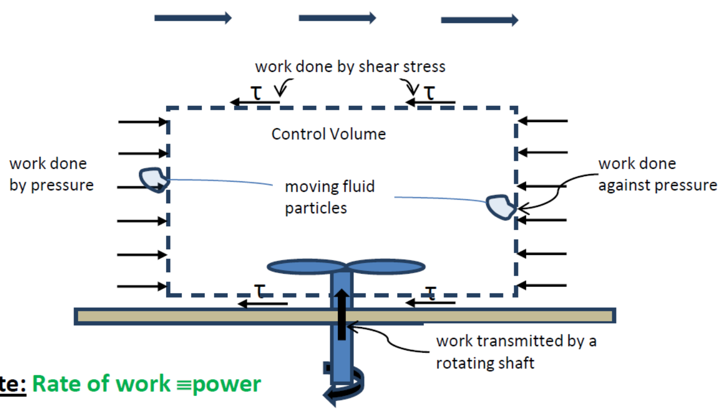
$$\left[\dot{Q} - \dot{W}\right]_{system} = \left[\dot{Q} - \dot{W}\right]_{control\ volume} \qquad (4)$$

Substituting Eqs. (3) and (4) into Eq. (1), we obtain

$$\dot{Q} - \dot{W} = \frac{\partial}{\partial t}\int_{C\forall} e\rho d\forall + \int_{CS} e\rho\vec{V}\cdot d\vec{A} \qquad (5)$$

$$\dot{Q} = ? \quad \dot{W} = ?$$

Rate of Work Done on a Control Volume



Note: Rate of work ≡power

If we neglect electrical and other equivalent forms of work, three types of work might be done on or by the fluid inside the control volume as shown in the figure above

1. Shaft Work ($\dot{W}_s$): is transmitted by a rotating shaft such as pump drive shaft or a turbine output shaft that is *"cut"* by the control surface. This work is done by shear stresses in the "cut" shaft, so it is somewhat similar to shear work. Shaft work is sometimes called **'pump work'** or **'turbine work'** if these devices are present.

2. Work Done on the Control Surface by Normal Stresses (Pressure Work):
Pressure work is done by fluid pressure acting on the boundaries of the control volume.

The work done by force $\vec{F}$ moved through distance ***ds*** is

$$\delta W = \vec{F} \cdot d\vec{s}$$

Rate of work $\quad \dot{W} = \lim_{\Delta t \to 0} \dfrac{\delta W}{\Delta t} = \lim_{\Delta t \to 0} \dfrac{\vec{F} \cdot d\vec{s}}{\Delta t} = \vec{F} \cdot \vec{V}$

The rate of work done on an element of area $d\vec{A}$ of the control surface by normal stress is given by

$$d\vec{F} \cdot \vec{V} = \sigma_{nn} d\vec{A} \cdot \vec{V}$$

The total rate of work done on the entire surface by normal stresses is given by (Since the work out across the boundaries of the control volume is the negative of the work done on the control volume)

$$\dot{W}_{normal} = \int_{CS} -\sigma_{nn} d\vec{A} \cdot \vec{V} = -\int_{CS} \sigma_{nn} \vec{V} \cdot d\vec{A}$$

$$Note: \sigma_{nn} = -p$$

Hence, $\dot{W}_{normal} = \int_{CS} p d\vec{A} \cdot \vec{V}$ (6)

3. Shear Work ($\dot{W}_{shear}$):
Shear work is done by shear stresses in the fluid acting on boundaries of the control volume. Similar to normal work,

$$\dot{W}_{shear} = -\int_{CS} \vec{\tau} \cdot \vec{V} d\vec{A}$$

Shear force acting on an area element dA is

$$d\vec{F} = \vec{\tau} dA$$

$\vec{\tau}$: shear stress acting in plane of dA

We often choose a control volume with control surfaces lying adjacent to solid boundaries, and with control surfaces cutting through inlet and outlet ports. Hence, the shear work can be expressed as two terms

$$\dot{W}_{shear} = -\int_{\substack{A(solid\\ surface)}} \vec{\tau} \cdot \vec{V} d\vec{A} - \int_{A(ports)} \vec{\tau} \cdot \vec{V} d\vec{A}$$

At solid surfaces $\vec{V}=0$, so the first term is zero (for a fixed control volume)

The last term can be made zero by proper choice of control surfaces. If we choose a control surface that cuts across each port perpendicular to the flow, then ***dA* is parallel to *V*** and hence, ***τ* is perpendicular to *V***. Thus, for control surfaces perpendicular to ***V***

$$\vec{\tau}\cdot\vec{V}=0 \quad \text{and} \quad \dot{W}_{shear}=0$$

Hence, energy equation for a control volume becomes

$$\dot{Q}-\dot{W}_s-\int_{CS} p\vec{V}\cdot d\vec{A}-\dot{W}_{shear}-\dot{W}_{other}=\frac{\partial}{\partial t}\int_{C\forall} e\rho d\forall+\int_{CS} e\rho\vec{V}\cdot d\vec{A}$$

where

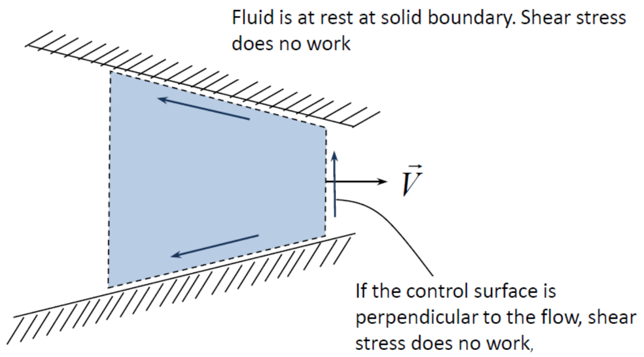
$$e=u+\frac{v^2}{2}+gz, \qquad\qquad u+p\vartheta=h$$

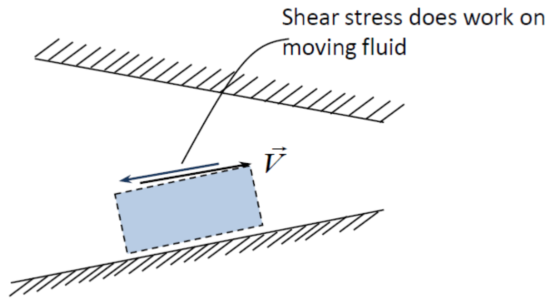
Substituting

$$\int_{CS} p\vec{V}\cdot d\vec{A}=\int_{CS} p\vartheta\rho\vec{V}\cdot d\vec{A}$$

$$\dot{Q}-\dot{W}_s-\dot{W}_{shear}-\dot{W}_{other}=\frac{\partial}{\partial t}\int_{C\forall} e\rho d\forall+\int_{CS}\left(e+p\vartheta\right)\rho\vec{V}\cdot d\vec{A}$$

Note: $\dot{W}_{shear}$ is zero if there is no control surface that lies within a moving fluid





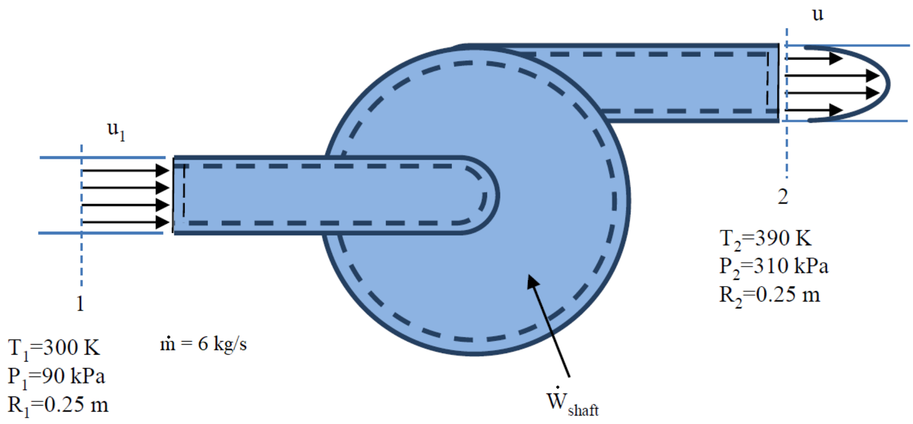
Example: A compressor compresses ***6 kg/s*** of air from inlet conditions ***T1 = 300 K*** and ***P1 = 90 kPa*** to discharge conditions ***T2 = 390 K* and *P2 = 310 kPa***. The air in the inlet pipe has a uniform velocity profile. The air in the discharge pipe has a parabolic velocity profile given by

$$u = u_{\max}\left[1-\left(\frac{r}{R_2}\right)^2\right]$$

where ***R2*** is the inside radius of the of the discharge pipe. Elevation changes are negligible, and the internal energy change of the air is given by

$$u_2 - u_1 = C_v\left(T_2 - T_1\right)$$

Assuming steady flow and negligible heat transfer, find the power required to drive the compressor.



Assumptions:
- Elevation changes are negligible
- Steady flow
- Heat transfer is negligible

Basic equation: Energy equation:

$$\dot{Q} - \dot{W}_s - \dot{W}_{shear} - \dot{W}_{other} = \frac{\partial}{\partial t}\int_{C\forall} e\rho d\forall + \int_{CS}\left(e + p\vartheta\right)\rho\vec{V}\cdot d\vec{A}$$

Simplifying the energy equation to the assumptions we obtain,

$$\underset{=0}{\dot{Q}} - \dot{W}_s - \underset{=0}{\dot{W}_{shear}} - \underset{=0}{\dot{W}_{other}} = \underbrace{\frac{\partial}{\partial t}\int_{C\forall} e\rho d\forall}_{=0} + \int_{CS}\left(e + p\vartheta\right)\rho\vec{V}\cdot d\vec{A}$$

$$\dot{W}_s = -\int_{CS}\left(e + p\vartheta\right)\rho\vec{V}\cdot d\vec{A} \qquad e = u + \frac{V^2}{2} + \underset{=0}{gz}$$

$$\dot{W}_s = -\int_{CS}\left[\left(u+\frac{V^2}{2}\right)+p\vartheta\right]\rho\vec{V}\cdot d\vec{A} = -\int_{A_{in}}\left[\left(u+\frac{V^2}{2}\right)+p\vartheta\right]\rho\vec{V}\cdot d\vec{A} - \int_{A_{out}}\left[\left(u+\frac{V^2}{2}\right)+p\vartheta\right]\rho\vec{V}\cdot d\vec{A}$$

Assuming that temperature and pressure are uniform at the inlet and outlet, we get

$$\dot{W}_s = -\int_{CS}\left[\left(u+\frac{V^2}{2}\right)+p\vartheta\right]\rho\vec{V}\cdot d\vec{A} = -\int_{A_{in}}\left[\left(u+\frac{V^2}{2}\right)+p\vartheta\right]\rho\vec{V}\cdot d\vec{A} - \int_{A_{out}}\left[\left(u+\frac{V^2}{2}\right)+p\vartheta\right]\rho\vec{V}\cdot d\vec{A}$$

$$\dot{W}_s = -\left[\left(u_1+\frac{V_1^2}{2}\right)+p_1\vartheta_1\right]\left\{-\left|\rho V_1 A_1\right|\right\} - \int_{A_{out}}\left[u+p\vartheta\right]\rho\vec{V}\cdot d\vec{A} - \int_{A_{out}}\frac{V^2}{2}\rho\vec{V}\cdot d\vec{A}$$

$$\dot{W}_s = -\left[\left(u_1+\frac{V_1^2}{2}\right)+p_1\vartheta_1\right]\left\{-\left|\rho V_1 A_1\right|\right\} - \int_{A_{out}}\left[u+p\vartheta\right]\rho\vec{V}\cdot d\vec{A} - \int_{A_{out}}\frac{V^2}{2}\rho\vec{V}\cdot d\vec{A}$$

$$\dot{W}_s = -\left(u_1+\frac{V_1^2}{2}+p_1\vartheta_1\right)\left(-\dot{m}\right)+\left(u_2+p_2\vartheta_2\right)\left\{\left|\rho V_2 A_2\right|\right\} - \int_{A_{out}}\rho\frac{V^3}{2}dA$$

$$\int_{A_{out}}\rho\frac{V^3}{2}dA = \int_0^{R_2}\rho\frac{V^3}{2}2\pi r dr = 2\pi\int_0^{R_2}\rho\frac{u_{\max}^3}{2}\left[1-\left(\frac{r}{R_2}\right)^2\right]^3 r dr = \frac{\pi\rho_2 u_{\max}^3 R_2^2}{8}$$

We relate $u_{\max}$ to the average velocity by,

$$\dot{m} = \rho_2\bar{V}_2 A_2 = \int\rho_2 u_2 dA = \int_0^{R_2}\rho_2 u_{\max}\left[1-\left(\frac{r}{R_2}\right)^2\right]2\pi r dr$$

$$\rho_2\bar{V}_2\pi R_2^2 = \rho_2\frac{u_{\max}}{2}\pi R_2^2 \quad\Rightarrow\quad u_{\max} = 2\bar{V}_2$$

$$\int_{A_{out}}\rho\frac{V^3}{2}dA = \frac{\pi\rho_2 u_{\max}^3 R_2^2}{8} = \frac{\pi\rho_2\left(2\bar{V}_2\right)^3 R_2^2}{8} = \pi\rho_2\bar{V}_2^3 R_2^2 = \dot{m}\bar{V}_2^2$$

$$\dot{W}_s = -\dot{m}\left(u_2-u_1\right)-\dot{m}\left(p_2\vartheta_2-p_1\vartheta_1\right)-\dot{m}\left(\bar{V}_2^2-\frac{V_1^2}{2}\right)$$

Assuming air is an ideal gas,

$$u_2-u_1 = C_v\left(T_2-T_1\right)$$

$$p_2\vartheta_2-p_1\vartheta_2 = R\left(T_2-T_1\right)$$

Also,

$$\left.\begin{aligned} \bar{V}_2 &= \frac{\dot{m}}{\rho_2 A_2} \\ \rho_2 &= \frac{p_2}{RT_2} \end{aligned}\right\} \Rightarrow \bar{V}_2 = \frac{\dot{m}RT_2}{p_2 A_2}$$

$$\bar{V}_2 = \frac{6\left[\frac{kg}{s}\right]287\left[\frac{Nm}{kgK}\right]390\left[K\right]}{310000\left[\frac{N}{m^2}\right]0.196\left[m^2\right]} = 11\left[\frac{m}{s}\right]$$

$$V_1 = \frac{\dot{m}}{\rho_1 A_1} = \frac{\dot{m}RT_1}{p_1 A_1} = \frac{6\left[\frac{kg}{s}\right]287\left[\frac{Nm}{kgK}\right]300\left[K\right]}{90000\left[\frac{N}{m^2}\right]0.196\left[m^2\right]} = 29.3\left[\frac{m}{s}\right]$$

Substituting back into expression for $\dot{W}_s$

$$\dot{W}_s = -\dot{m}\left(u_2 - u_1\right) - \dot{m}\left(p_2\vartheta_2 - p_1\vartheta_1\right) - \dot{m}\left(\bar{V}_2^2 - \frac{V_1^2}{2}\right)$$

$$\dot{W}_s = -6\left[\frac{kg}{s}\right]720\left[\frac{j}{kgK}\right]\left(390-300\right)\left[K\right] - 6\left[\frac{kg}{s}\right]2807\left[\frac{j}{kgK}\right]\left(390-300\right)\left[K\right] - 6\left[\frac{kg}{s}\right]\left(11.2^2 - 29.3^2\right)$$

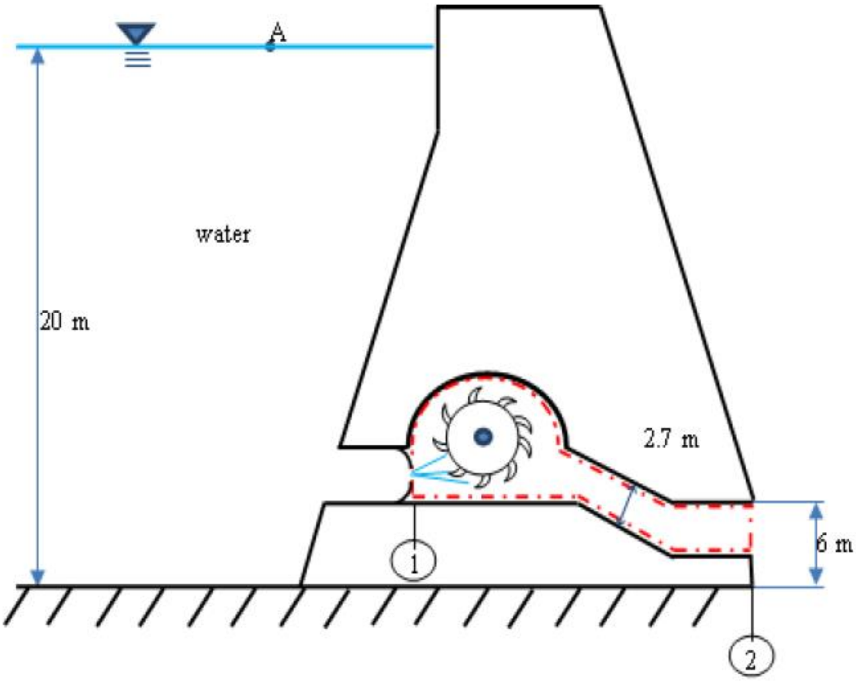
$$\dot{W}_s = -\left(388800 + 155000 - 1800\right)\left[\frac{j}{s}\right]$$

$$\dot{W}_s = -\left(\underbrace{388800}_{\text{internal energy}} + \underbrace{155000}_{\substack{\text{flow work} \\ \text{"mechanical} \\ \text{pressure enrgy"}}} - \underbrace{1800}_{\text{kinetic energy}}\right)\left[\frac{j}{s}\right]$$

$$\dot{W}_s = -542\left[kW\right]$$

Note that a large portion of the compressor input work appears as an increase in the thermal (internal) energy and the mechanical “pressure energy” of this comnpressible fluid. The kinetic energy change is much smaller.

Example: Turbines convert the energy contained within a fluid into mechanical energy or shaft work. A turbine is installed in a dam as shown in the figure. Water is permitted to flow through a passage way to the turbine after which the water drains downstream. For the data given in the figure, determine the power available to the turbine when the discharge at the outlet is **30 m³/s**.



Assumptions:
- Steady flow
- Incompressible flow
- No heat transfer
- Internal energy change can be neglected

Basic equation: Energy equation:

$$\dot{Q}-\dot{W}_s-\dot{W}_{shear}-\dot{W}_{other}=\frac{\partial}{\partial t}\int_{C\forall} e\rho d\forall+\int_{CS}(e+p\vartheta)\rho\vec{V}\cdot d\vec{A}$$

With these assumptions, energy equation becomes

$$\dot{W}_s=-\int_{CS}(e+p\vartheta)\rho\vec{V}\cdot d\vec{A}$$

$$\dot{W}_s=-\int_{A_1}(e+p\vartheta)\rho\vec{V}\cdot d\vec{A}-\int_{A_2}(e+p\vartheta)\rho\vec{V}\cdot d\vec{A}$$

$e=u+\dfrac{V^2}{2}+gz$ assuming also flow is uniform

$$\dot{W}_s = -\left[(e+p\vartheta)\left\{-|\rho VA|\right\}\right]_1 - \left[(e+p\vartheta)\left\{|\rho VA|\right\}\right]_2$$
$$= -\left[(e+p\vartheta)(-\dot{m})\right]_1 - \left[(e+p\vartheta)\dot{m}\right]_2$$
$$= -\left[\left(u_2 + \frac{V_2^2}{2} + gz_2 + p_2\vartheta_2\right) - \left(u_1 + \frac{V_1^2}{2} + gz_1 + p_1\vartheta_1\right)\right]\dot{m}$$
$$= -\left[(u_2 - u_1) + \frac{V_2^2 - V_1^2}{2} + (p_2\vartheta_2 - p_1\vartheta_1) + g(z_2 - z_1)\right]\dot{m}$$

Note: $u_2 - u_1 \approx 0$ negligible internal energy change.

$\vartheta_1 = \vartheta_2$ incompressible flow

$$\dot{W}_s = -\dot{m}\left[\frac{V_2^2}{2} - \frac{V_1^2}{2} + g(z_2 - z_1)\right]$$

V_1 can be found using Bernoulli equation, between free surface and nozzle exit

$$\frac{p_A}{\rho} + \frac{V_A^2}{2} + gz_A = \frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1$$
$$p_1 \cong p_A = p_{atm}$$
$$V_A \cong 0$$
$$gz_A = \frac{V_1^2}{2} + gz_1 \quad \Rightarrow \quad \frac{V_1^2}{2} = g(z_A - z_1)$$

$$\dot{W}_s = -\dot{m}\left[\frac{V_2^2}{2} - g(z_A - z_1) + g(z_2 - z_1)\right]$$
$$\dot{W}_s = -\dot{m}\left[\frac{V_2^2}{2} - g(z_A - z_2)\right]$$

From continuity equation

$$\dot{Q} = V_1A_1 = V_2A_2 \Rightarrow V_2 = \frac{\dot{Q}}{A_2} = \frac{30}{\pi\frac{(2.7)^2}{4}} = 5.24\left[\frac{m}{s}\right]$$

Substituting these values into energy equation,

$$\dot{W}_s = -\dot{m}\left[\frac{V_2^2}{2} - g\left(z_A - z_2\right)\right]$$

$$\dot{W}_s = -\rho\dot{Q}\left[\frac{V_2^2}{2} - g\left(z_A - z_2\right)\right]$$

$$\dot{W}_s = -(1000)(30)\left[\frac{5.24^2}{2} - 9.81(20-6)\right]$$

$\dot{W}_s = +3708336\left[\frac{Nm}{s}\right] \cong 3.7[MW]$ plus sign indicates that work is done by the system.

Average Properties and Velocities

Usually, the uniform flow assumption is only an approximation and we use average velocities and property values to calculate flow of energy at various inlet and outlet planes. For accurate calculations, we must carefully define the averages so that they truly represent associated energy flows. In most cases, appropriate average values of *u, p,* ρ and *z* are readily apparent because these properties are often closely uniform across the section. However, determination of average velocity and hence kinetic energy flux requires a careful attention.

Average Velocity is defined as $\bar{V} = \dfrac{\int_A \vec{V}\cdot d\vec{A}}{A}$

Average Kinetic Energy

Since velocity is usually non-uniform, representing the kinetic energy flux in terms of uniform velocity slightly more complicated.

$$\dot{E}_k = \int_A \rho \frac{V^2}{2}\vec{V}\cdot d\vec{A}$$

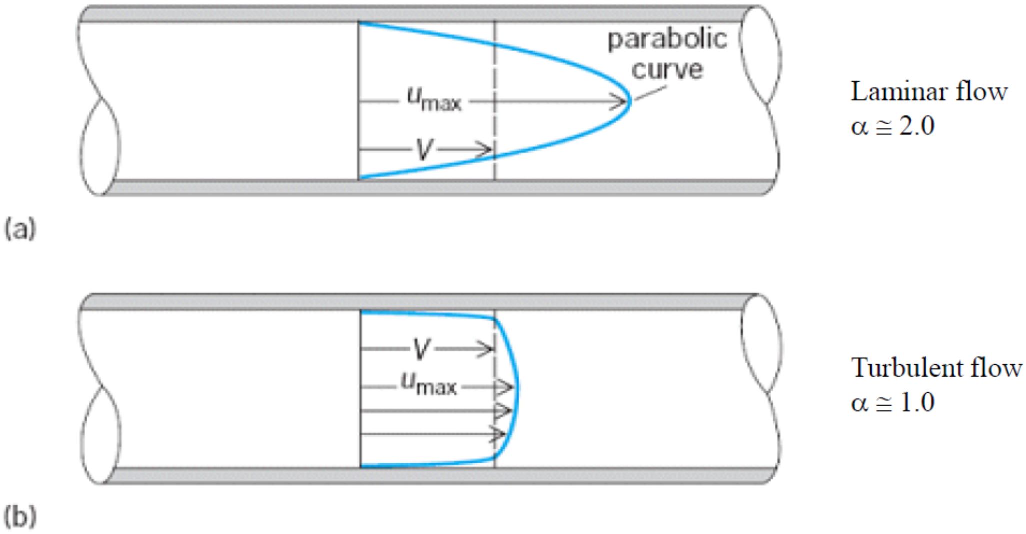
Since velocity is not uniform, $\dot{E}_k \neq \rho\dfrac{\bar{V}^2}{2}\bar{V}A$

hence, a kinetic energy correction factor α is defined by

$$\alpha = \frac{\int \rho \frac{V^3}{2} dA}{\rho \frac{\bar{V}^3}{2} A}$$

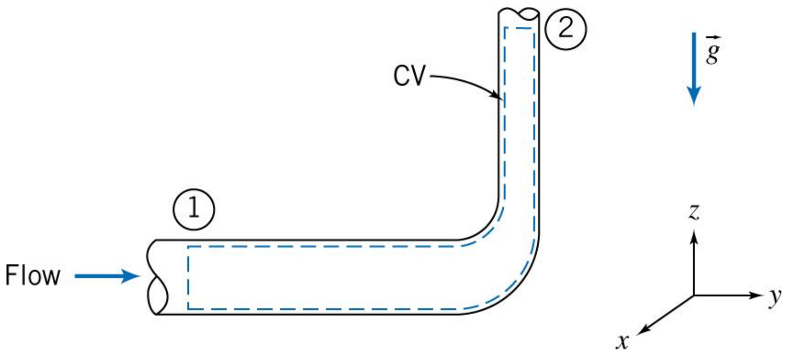
The true kinetic energy flux across a plane is $\dot{E}_k = \alpha\left(\dfrac{1}{2}\rho V^3 A\right)$

For flow in a circular pipe, α ranges from 2 for fully developed laminar flow to about 1.05 for fully developed turbulent flow.



INTERNAL INCOMPRESSIBLE VISCOUS FLOW

<u>Energy Equation for a Flow in a Pipe</u>



Assumptions
1. No shaft work
2. Incompressible flow
3. Steady flow
4. Internal energy and pressure are uniform at cross sections 1 and 2

$$\dot{Q} - \underset{=0}{\dot{W}_s} - \underset{=0}{\dot{W}_{shear}} - \underset{=0}{\dot{W}_{other}} = \underbrace{\frac{\partial}{\partial t}\int_{C\forall} e\rho d\forall}_{=0} + \int_{CS} \left(e + p\vartheta\right)\rho\vec{V}\cdot d\vec{A}$$

$$e = u + \frac{V^2}{2} + gz$$

Considering, u, p and ρ are uniform over inlet and outlet cross-sections, we can write

$$\dot{Q} = \dot{m}\left(u_2 - u_1\right) + \dot{m}\left(\frac{p_2}{\rho} - \frac{p_1}{\rho}\right) + \dot{m}\left(z_2 - z_1\right) + \int_{A_2}\frac{V_2^2}{2}\rho V_2 dA_2 - \int_{A_1}\frac{V_1^2}{2}\rho V_1 dA_1$$

<u>Note:</u> At cross-sections 1 and 2, velocity profiles are non-uniform. However, integrals in the above equation can be expressed in terms of average velocity and kinetic energy correction factor, i.e.

$$\int_{A_2}\frac{V^2}{2}\rho V dA = \alpha\frac{\bar{V}^2}{2}\rho\bar{V}A = \alpha\frac{\bar{V}^2}{2}\dot{m}$$

Therefore, energy equation becomes,

$$\dot{Q} = \dot{m}\left(u_2 - u_1\right) + \dot{m}\left(\frac{p_2}{\rho} - \frac{p_1}{\rho}\right) + \dot{m}\left(z_2 - z_1\right) + \dot{m}\left(\alpha_2\frac{\bar{V}_2^2}{2} - \alpha_1\frac{\bar{V}_1^2}{2}\right)$$

dividing by mass flow rate and rearranging, we get

$$\left(\frac{p_1}{\rho g}+\alpha_1\frac{\bar{V}_1^2}{2}+z_1\right)=\left(\frac{p_2}{\rho g}+\alpha_2\frac{\bar{V}_2^2}{2}+z_2\right)+\frac{u_2-u_1}{g}-\frac{\delta Q}{g\delta m}$$

$\frac{\dot{Q}}{\dot{m}}$ heat transfer rate per unit mass of moving fluid.

For incompressible flow *(combining the first and the second law of thermodynamics),*

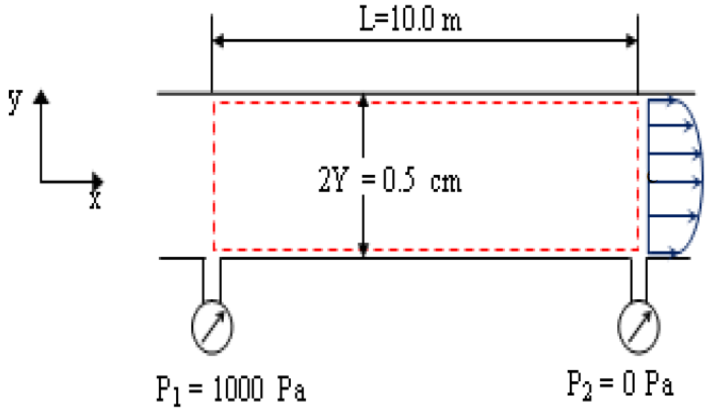
$$gh_f = u_2 - u_1 - q$$

Here h_f represents the loss of potential to perform useful work. It shows us that the internal energy (and hence temperature) of an incompressible fluid can be increased by two ways: heat transfer to the fluid and friction. Only one effect can cause an internal energy decrease; namely heat transfer from fluid, as gh_f cannot be negative.

$$\left(\frac{p_1}{\rho g}+\alpha_1\frac{\bar{V}_1^2}{2}+z_1\right)=\left(\frac{p_2}{\rho g}+\alpha_2\frac{\bar{V}_2^2}{2}+z_2\right)+h_f$$ **EXTENDED BERNOULLI EQUATION**

h_f is called head loss

Example:



$\rho = 850\ \text{kg/m}^3$

An incompressible viscous fluid flows between two horizontal parallel plates as shown. The plates are spaced **0.5 cm** apart and are very wide perpendicular to page. Flow is laminar and velocity profile at any cross section is given by,

$$u = -\frac{Y^2}{2\mu}\left(\frac{\Delta P}{L}\right)\left[1-\left(\frac{y}{Y}\right)^2\right]$$

where ΔP is the pressure change that occurs in length L. Calculate average mechanical energy loss h_f between the pressure gages. Then show that mechanical energy loss also satisfies the equations

$$gh_f = \frac{\int_{CV} \mu\left(\frac{\partial u}{\partial y}\right)^2 d\forall}{\dot{m}} \quad \text{and} \quad gh_f = \mathrm{Re}\left(\frac{L}{Y}\right)\left(\frac{\bar{V}^2}{2}\right) \quad \text{where } \mathrm{Re} = \frac{\rho \bar{V} Y}{\mu}$$

Using the energy equation (extended Bernoulli equation)

$$\left(\frac{p_1}{\rho g} + \alpha_1\frac{\bar{V}_1^2}{2} + z_1\right) = \left(\frac{p_2}{\rho g} + \alpha_2\frac{\bar{V}_2^2}{2} + z_2\right) + h_f$$

$$z_1 = z_2$$

Note: Velocity profiles are identical.

$$\therefore \qquad h_f = \frac{p_1 - p_2}{\rho g}$$

$$h_f = \frac{1000-0}{850(9.81)} = 0.119[m]$$

$$gh_f = \frac{p_1 - p_2}{\rho} = \frac{1000-0}{850} = 1.18\left[\frac{j}{kg}\right]$$

$$u = -\frac{Y^2}{2\mu}\left(\frac{\Delta P}{L}\right)\left[1-\left(\frac{y}{Y}\right)^2\right]$$

$$\frac{\partial u}{\partial y} = \frac{y}{\mu}\frac{\Delta P}{L}$$

$$\frac{\int_{CV} \mu\left(\frac{\partial u}{\partial y}\right)^2 d\forall}{\dot{m}} = \frac{2\mu\int_0^Y \left[\frac{y}{\mu}\frac{\Delta P}{L}\right]^2 WL\,dy}{\rho\bar{V}\,2YW} = \frac{L\left(\frac{\Delta P}{L}\right)^2}{\rho\mu\bar{V}Y}\int_0^Y y^2 dy = \frac{Y^2 L}{3\rho\mu\bar{V}}\left(\frac{\Delta P}{L}\right)^2$$

$$\bar{V} = \frac{2\int_0^Y u\,dy}{2Y} = \frac{-2\int_0^Y \frac{Y^2}{2\mu}\left(\frac{\Delta P}{L}\right)\left[1-\left(\frac{y}{Y}\right)^2\right]dy}{2Y} = -\frac{1}{3}\left(\frac{Y^2}{\mu}\right)\left(\frac{\Delta P}{L}\right) = \frac{2}{3}u_{\max}$$

$\therefore$

$$\frac{\int_{CV} \mu \left(\frac{\partial u}{\partial y}\right)^2 d\forall}{\dot{m}} = -\frac{1}{\rho}\left(\frac{\Delta P}{L}\right) L = \frac{P_1 - P_2}{\rho} = gh_f$$

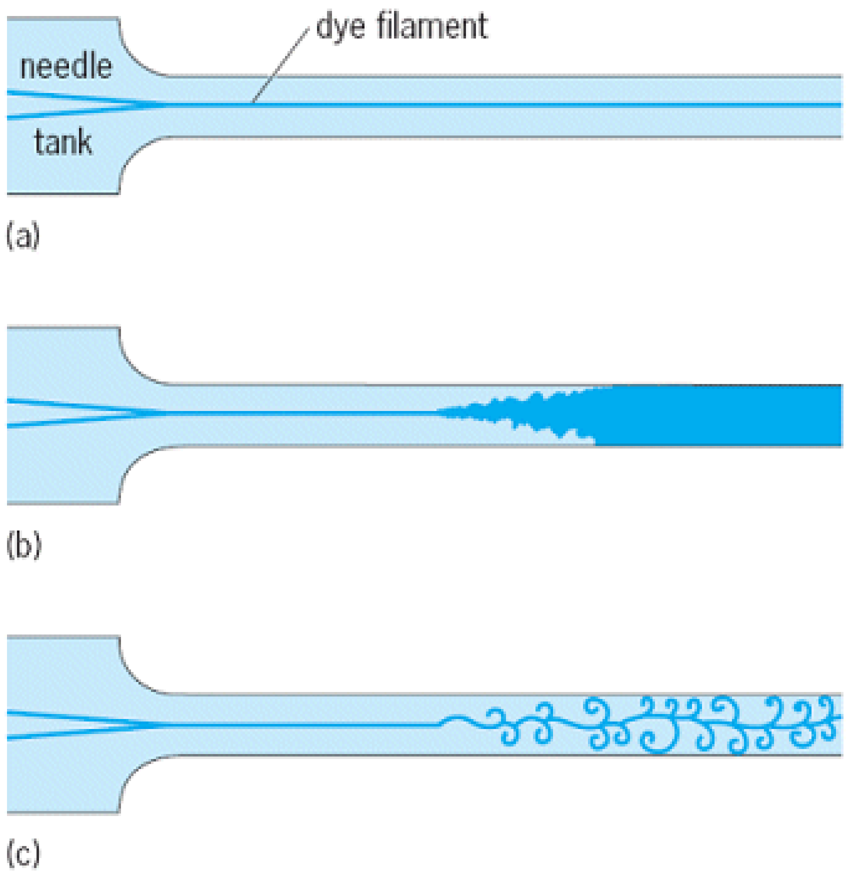
CLASSIFYING THE FLOW IN A PIPE OR DUCT

Laminar and Turbulent Flow

If the flow in a pipe is **laminar**, the fluid moves along smooth streamlines.

If the flow is **turbulent**, a rather violent mixing of the fluid occurs, and the fluid velocity at a point varies randomly with time.

The difference between laminar and turbulent flows were classified by Osborne Reynolds in 1883. Reynolds performed a series of experiments.



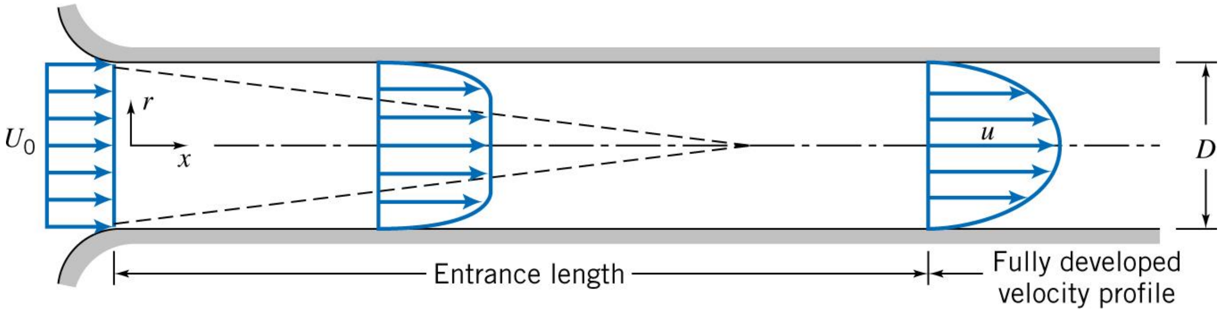
Pipe-flow transition experiment. (a) laminar flow. (b) High ReD, turbulent flow. (c) Spark photograph of turbulent flow condition. (After O. Reynolds, an experimental investigation of the circumstances which determine whether the motion of water shall be direct or sinuous and of the law of resistance in parallel channels, Phil. Trans. Roy. Soc., London, A174:935–982, 1883)

Reynolds' experiments showed that the nature of the pipe flow depends on the Reynolds number,

$$\mathrm{Re} = \frac{\rho \bar{V} d}{\mu}$$

Developing and Fully Developed Flow

The flow in a constant area duct or pipe is said to be fully developed if the shape of the velocity profile is the same at all cross sections.



The length L_e is called entrance length or the developing length.

$$\frac{L_e}{d} = f\left(\mathrm{Re}\right)$$

$$\frac{L_e}{d} \cong 0.06\,\mathrm{Re} \quad \text{laminar flow}$$

$$\frac{L_e}{d} \cong 4.4\left(\mathrm{Re}\right)^{1/6} \text{ turbulent flow}$$