# USE OF THE MOMENTUM INTEGRAL EQUATION FOR ZERO PRESSURE GRADIENT FLOW

For special case of flow over a flat plate, $U = \text{constant}$. From Bernoulli's equation we see that for this case, $P = \text{constant}$ and, thus $dP/dx = 0$

The momentum integral equation then reduces to

$$\tau_w = \rho U^2 \frac{d\theta}{dx} = \rho U^2 \frac{d}{dx} \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$$

If new integration of variable is defined as

$$\eta = \frac{y}{\delta} \qquad \text{then} \qquad dy = \delta\, d\eta$$

and momentum integral equation for zero pressure gradient is written

$$\tau_w = \rho U^2 \frac{d\theta}{dx} = \rho U^2 \frac{d\delta}{dx} \int_0^1 \frac{u}{U}\left(1 - \frac{u}{U}\right) d\eta$$

In order to solve this equation for boundary-layer thickness, we must

1) Assume a velocity profile in the form

$$\frac{u}{U} = f\left(\frac{y}{\delta}\right)$$

2) Assumed velocity profile should satisfy the following boundary conditions:

at $y=0$ $\qquad u=0$

at $y=\delta$ $\qquad u=U$

at $y=\delta$ , $\qquad \frac{\partial u}{\partial y}=0$

at $y=0$ $\qquad \frac{\partial^2 u}{\partial y^2}=0$

3) Obtain

$$\beta = \frac{\theta}{\delta} = \int_0^1 \frac{u}{U}\left(1-\frac{u}{U}\right) d\eta \quad = \text{constant}$$

4) The momentum integral equation becomes

$$\tau_w = \rho U^2 \frac{d\delta}{dx} \beta = \mu \frac{du}{dy}\Big|_{y=0}$$

5)

<u>Example</u>: Consider laminar flow over a flat plate. Assume the velocity profile in the form

$$\frac{u}{U} = a + b\left(\frac{y}{\delta}\right) + c\left(\frac{y}{\delta}\right)^2 + d\left(\frac{y}{\delta}\right)^3 \qquad \eta = \frac{y}{\delta}$$

a) Find the variation of the boundary layer thickness

b) Determine the wall shear stress coefficient, $C_f$.

a) at $y=0$ $\qquad u=0 \quad \Rightarrow \quad a=0$

at $y=\delta$ $\qquad u=U \quad \Rightarrow \quad 1 = b+c+d$

$$\left.\begin{array}{ll} \text{at } y=\delta & \frac{\partial u}{\partial y}=0 \\ (\text{at } \eta \to 1 & \frac{\partial u}{\partial \eta}=0 \end{array}\right\} \Rightarrow \quad 0 = b + 2c + 3d$$

$$\left.\begin{array}{ll} \text{at } y=0 & \dfrac{\partial^2 u}{\partial y^2}=0 \\ (\text{at } \eta=0 & \dfrac{\partial^2 u}{\partial \eta^2}=0 \end{array}\right\} \Rightarrow \quad 0 = 2c \quad \Rightarrow \quad c=0$$

$\therefore$

$$\begin{array}{rl} 1 = & b + d \\ - \; 0 = & b + 3d \\ \hline 1 = & -2d \end{array} \quad \Rightarrow \quad d = -\frac{1}{2} \quad \text{and} \quad b = \frac{3}{2}$$

$\therefore$ Velocity profile becomes

$$\frac{u}{U} = \frac{3}{2}\eta - \frac{1}{2}\eta^3 \qquad \eta = \frac{y}{\delta}$$

The wall shear stress is

$$\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} = \mu \left.\frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial y}\right|_{\eta=0} = \mu\left(\frac{3}{2} - \frac{3}{2}\eta^2\right)\left.\frac{U}{\delta}\right|_{\eta=0} = \frac{3\mu U}{2\delta}$$

We are now in a position to apply the momentum integral equation

$$\tau_w = \rho U^2 \frac{d\delta}{dx}\int_0^1 \frac{u}{U}\left(1 - \frac{u}{U}\right) d\eta$$

$$\frac{3\mu U}{2\delta} = \rho U^2 \frac{d\delta}{dx}\int_0^1 \left(\frac{3}{2}\eta - \frac{1}{2}\eta^3\right)\left(1 - \frac{3}{2}\eta + \frac{1}{2}\eta^3\right) d\eta$$

or; this equation may be rearranged as

$$\frac{3\mu U}{2\delta \rho U^2} = \frac{d\delta}{dx}\int_0^1 \left( \frac{3}{2}\eta - \frac{9}{4}\eta^2 - \frac{1}{2}\eta^3 + \frac{3}{2}\eta^4 - \frac{1}{4}\eta^6 \right) d\eta$$

Integrating and substituting limits yields

$$\frac{3\mu}{2\delta\rho U} = \frac{39}{280}\frac{d\delta}{dx}$$

or

$$\delta\, d\delta = \frac{140\,\mu}{13\,\rho U}\, dx$$

integrating it

$$\frac{\delta^2}{2} = \frac{140\,\mu}{13\,\rho U}\, x + c$$

if it is assumed that $\delta = 0$ at $x = 0$, then $c = 0$ and thus

$$\delta = \sqrt{\frac{280\,\mu\, x}{13\rho U}} \Rightarrow \frac{\delta}{x} = \frac{4.641}{\sqrt{Re_x}}$$

b) The wall shear stress coefficient, $C_f$

$$C_f \equiv \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{\frac{3\mu U}{2\delta}}{\frac{1}{2}\rho U^2} = \frac{3\mu}{\rho U \delta} = 3\,\frac{\mu}{\rho U x}\,\frac{x}{\delta}$$

$$C_f = \frac{3}{Re_x}\cdot\frac{\sqrt{Re_x}}{4.641} = \frac{0.646}{\sqrt{Re_x}}$$

## EXAMPLE 9.3—Laminar Boundary Layer on a Flat Plate: Approximate Solution Using Sinusoidal Velocity Profile

Consider two-dimensional laminar boundary-layer flow along a flat plate. Assume the velocity profile in the boundary layer is sinusoidal,

$$\frac{u}{U} = \sin\left(\frac{\pi}{2}\frac{y}{\delta}\right)$$

Find expressions for:

(a) the rate of growth of $\delta$ as a function of $x$.
(b) the displacement thickness, $\delta^*$, as a function of $x$.
(c) the total friction force on a plate of length $L$ and width $b$.

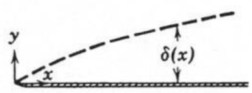
### EXAMPLE PROBLEM 9.3

GIVEN: Two-dimensional, laminar boundary-layer flow along a flat plate. The boundary-layer velocity profile is

$$\frac{u}{U} = \sin\left(\frac{\pi}{2}\frac{y}{\delta}\right) \qquad \text{for } 0 \leq y \leq \delta$$

and

$$\frac{u}{U} = 1 \qquad \text{for } y > \delta$$




FIND: (a) $\delta(x)$.
(b) $\delta^*(x)$.
(c) Total friction force on a plate of length $L$ and width $b$.

SOLUTION:
For flat plate flow, $U$ = constant, $dp/dx = 0$, and

$$\tau_w = \rho U^2 \frac{d\theta}{dx} = \rho U^2 \frac{d\delta}{dx} \int_0^1 \frac{u}{U}\left(1 - \frac{u}{U}\right) d\eta \qquad (9.19)$$

Assumptions: (1) Steady flow
(2) Incompressible flow

Substituting $\frac{u}{U} = \sin\frac{\pi}{2}\eta$ into Eq. 9.19, we obtain

$$\tau_w = \rho U^2 \frac{d\delta}{dx} \int_0^1 \sin\frac{\pi}{2}\eta\left(1 - \sin\frac{\pi}{2}\eta\right) d\eta$$

$$= \rho U^2 \frac{d\delta}{dx} \int_0^1 \left(\sin\frac{\pi}{2}\eta - \sin^2\frac{\pi}{2}\eta\right) d\eta$$

$$= \rho U^2 \frac{d\delta}{dx}\frac{2}{\pi}\left[-\cos\frac{\pi}{2}\eta - \frac{1}{2}\frac{\pi}{2}\eta + \frac{1}{4}\sin\pi\eta\right]_0^1$$

$$= \rho U^2 \frac{d\delta}{dx}\frac{2}{\pi}\left[0 + 1 - \frac{\pi}{4} + 0 + 0 - 0\right]$$

$$\tau_w = 0.137\rho U^2 \frac{d\delta}{dx} = \beta\rho U^2 \frac{d\delta}{dx}; \quad \beta = 0.137$$

Now

$$\tau_w = \mu \frac{\partial u}{\partial y}\bigg]_{y=0} = \mu \frac{U}{\delta}\frac{\partial(u/U)}{\partial(y/\delta)}\bigg]_{y=0} = \mu\frac{U}{\delta}\frac{\pi}{2}\cos\frac{\pi}{2}\eta\bigg]_{\eta=0} = \frac{\pi\mu U}{2\delta}$$

Therefore,

$$\tau_w = \frac{\pi \mu U}{2\delta} = 0.137 \rho U^2 \frac{d\delta}{dx}$$

Separating variables gives

$$\delta \, d\delta = 11.5 \frac{\mu}{\rho U} dx$$

Integrating, we obtain

$$\frac{\delta^2}{2} = 11.5 \frac{\mu}{\rho U} x + c$$

But $c = 0$, since $\delta = 0$ at $x = 0$, so

$$\delta = \sqrt{23.0 \frac{x \mu}{\rho U}}$$

or

$$\frac{\delta}{x} = 4.80 \sqrt{\frac{\mu}{\rho U x}} = \frac{4.80}{\sqrt{Re_x}} \longleftarrow \delta(x)$$

The displacement thickness, $\delta^*$, is given by

$$\delta^* = \delta \int_0^1 \left(1 - \frac{u}{U}\right) d\eta$$

$$= \delta \int_0^1 \left(1 - \sin \frac{\pi}{2} \eta\right) d\eta = \delta \left[\eta + \frac{2}{\pi} \cos \frac{\pi}{2} \eta\right]_0^1$$

$$\delta^* = \delta \left[1 - 0 + 0 - \frac{2}{\pi}\right] = \delta \left[1 - \frac{2}{\pi}\right]$$

Since, from part (a),

$$\frac{\delta}{x} = \frac{4.80}{\sqrt{Re_x}}$$

then

$$\frac{\delta^*}{x} = \left(1 - \frac{2}{\pi}\right) \frac{4.80}{\sqrt{Re_x}} = \frac{1.74}{\sqrt{Re_x}} \longleftarrow \delta^*(x)$$

The total friction force on one side of the plate is given by

$$F = \int_{A_p} \tau_w \, dA$$

Since $dA = b\,dx$ and $0 \le x \le L$,

$$F = \int_0^L \tau_w b \, dx = \int_0^L \rho U^2 \frac{d\theta}{dx} b \, dx = \rho U^2 b \int_0^{\theta_L} d\theta = \rho U^2 b \theta_L$$

$$\theta_L = \int_0^{\delta_L} \frac{u}{U}\left(1 - \frac{u}{U}\right) dy = \delta_L \int_0^1 \frac{u}{U}\left(1 - \frac{u}{U}\right) d\eta = \beta \delta_L$$

From part (a), $\beta = 0.137$ and $\delta_L = \dfrac{4.80L}{\sqrt{Re_L}}$, so

$$F = \frac{0.658 \rho U^2 bL}{\sqrt{Re_L}} \longleftarrow F$$

{This problem illustrates the application of the momentum integral equation to a flat plate, laminar, boundary-layer flow.}

# APPROXIMATE SOLUTION OF THE TURBULENT FLOW OVER A FLAT PLATE WITH ZERO PRESSURE GRADIENT

To start the approximate solution, it is necessary to assume a velocity profile in the form $u/U = f(y/\delta)$. For the assumed velocity profile, one may use the power law form which is given by

$$\frac{u}{U} = \left(\frac{y}{\delta}\right)^{n} = \eta^{n}$$

An exponent of $n = 1/7$ is typically used to model the velocity profile

$$\frac{u}{U} = \left(\frac{y}{\delta}\right)^{1/7} = \eta^{1/7}$$

However, this profile does not hold in the immediate vicinity of the wall, since at the wall it predicts $du/dy \to \infty$. Consequently, we cannot use this profile in the definition of $\tau_w$. For turbulent flow, $\tau_w$ is evaluated by the expression developed for pipe flow.

$$\tau_w = 0.03325\, \rho \bar{V}^2 \left[\frac{\nu}{R\bar{V}}\right]^{0.25}$$

For a $\frac{1}{7}$-power profile in a pipe $\bar{V}/U = 0.817$ and $R = \delta$. Substituting this values into above expression, one obtains

$$\tau_w = 0.0233\, \rho U^2 \left(\frac{\nu}{U\delta}\right)^{1/4}$$

The momentum integral equation for zero pressure gradient flow, is

$$\tau_w = \rho U^2 \frac{d\delta}{dx} \int_0^1 \frac{u}{U}\left(1-\frac{u}{U}\right) d\eta$$

Substituting $\tau_w$ and $u/U$ and integrating, we obtain

$$0.0233\left(\frac{\nu}{U\delta}\right)^{1/4} = \frac{d\delta}{dx}\int_0^1 \eta^{1/7}\left(1-\eta^{1/7}\right) d\eta = \frac{7}{72}\frac{d\delta}{dx}$$

Thus we obtain a differential equation for $\delta$:

$$\delta^{1/4}\, d\delta = 0.240\left(\frac{\nu}{U}\right)^{1/4} dx$$

Integrating gives

$$\frac{4}{5}\delta^{5/4} = 0.240\left(\frac{\nu}{U}\right)^{1/4} x + C$$

If it is assumed that $\delta \approx 0$ at $x=0$, then $C=0$, and

$$\delta = 0.382\left(\frac{\nu}{U}\right)^{1/5} x^{4/5}$$

or

$$\frac{\delta}{x} = 0.382\left(\frac{\nu}{Ux}\right)^{1/5} = \frac{0.382}{Re_x^{1/5}}$$

The skin friction coefficient, $C_f$

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = 0.0466\left(\frac{\nu}{U\delta}\right)^{1/4}$$

Substituting for $\delta$

$$C_f = \frac{0.0594}{Re_x^{1/5}}$$

Use of the momentum integral equation is an approximate technique to predict boundary-layer development; the equation predicts trends correctly. Parameters of laminar boundary layer vary as $Re_x^{-1/2}$; those for the turbulent boundary layer vary as $Re_x^{-1/5}$. The turbulent boundary layer develops more rapidly than the laminar boundary layer.

Example: Water with a density of 1000 kg/m³ and kinematic viscosity of $1 * 10^{-6}$ m²/sn is flowing over a flat plate. Water approaches to the flat plate at a uniform velocity of 0.5 m/sn. The flat plate is 8 m long and 2 m wide. Determine the friction drag on one side of the flat plate.

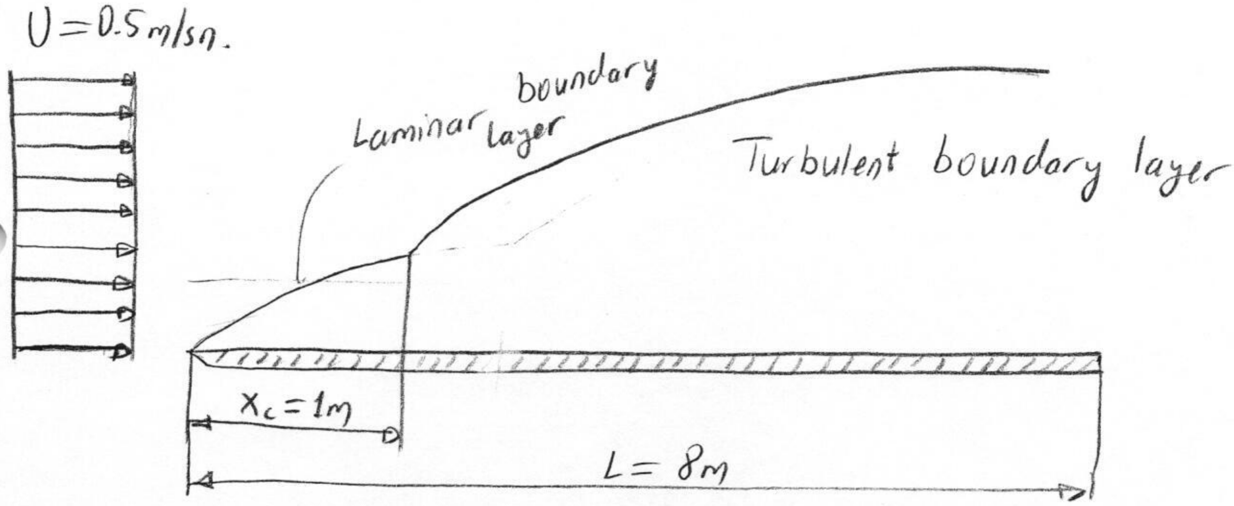
SOLUTION: In order to determine the friction drag on one side of the flat plate, one should first determine whether the flow over the flat plate is laminar or turbulent. For this reason, the critical length for transition from the laminar flow regime to the turbulent regime should be determined. Since,

$$Re_c = \frac{U L_c}{\nu} = 500{,}000$$

then the critical length, $L_c$, for transition is

$$L_c = \frac{\nu Re_c}{U} = \frac{1 \ast 10^{-6} \ast 500000}{0.5} = 1\,m.$$

Hence there exists both laminar and a turbulent boundary layer over the flat plate.




The drag force, D, on the flat plate may be evaluated as

$$D = D_{laminar} + D_{turbulent}$$

$$D = \int_0^{x_c} \tau_{w\,lam}\, b\, dx + \int_{x_c}^{L} \tau_{w\,tur}\, b\, dx$$

Since $\quad C_{f_{lam}} = \dfrac{\tau_{w\,lam}}{\frac{1}{2}\rho U^2} = \dfrac{0.664}{\sqrt{Re_x}}$

$$Re_x = \frac{Ux}{\nu}$$

$$C_{f_{tur}} = \frac{\tau_{w\,tur}}{\frac{1}{2}\rho U^2} = \frac{0.0594}{Re_x^{1/5}}$$

$$\tau_{w_{lam}} = \frac{0.664}{\sqrt{\frac{Ux}{\nu}}} \frac{1}{2} \rho U^2 = 0.332 \rho U^2 \left(\frac{\nu}{Ux}\right)^{1/2}$$

$$\tau_{w\,tur} = \frac{0.0594}{\left(\frac{Ux}{\nu}\right)^{1/5}} \frac{1}{2} \rho U^2 = 0.0297 \rho U^2 \left(\frac{\nu}{Ux}\right)^{1/5}$$

Therefore

$$D = \int_0^{X_c=1} 0.332 \rho U^2 \left(\frac{\nu}{Ux}\right)^{1/2} b\,dx + \int_{X_c=1}^{L=8} 0.0297 \rho U^2 \left(\frac{\nu}{Ux}\right)^{1/5} b\,dx$$

$$D = 0.332 \rho U^2 b \left(\frac{\nu}{U}\right)^{1/2} \left(2x^{1/2}\right)\Big|_0^1 +$$

$$+ 0.0297 \rho U^2 b \left(\frac{\nu}{U}\right)^{1/5} \left(\frac{5X^{4/5}}{4}\right)\Big|_1^8$$

$$D = 0.332 * 1000 * 0.5^2 * 2 * \left(\frac{10^{-6}}{0.5}\right)^{1/2} \left(2 * 1^{1/2}\right) +$$

$$+ 0.0297 * 1000 * 0.5^2 * 2 * \left(\frac{10^{-6}}{0.5}\right)^{1/5} * \frac{5}{4}\left[8^{4/5} - 1^{4/5}\right]$$

$$D = 0.47 + 5.756 = 6.226 \text{ N}$$