Sinyaller ve Sistemler

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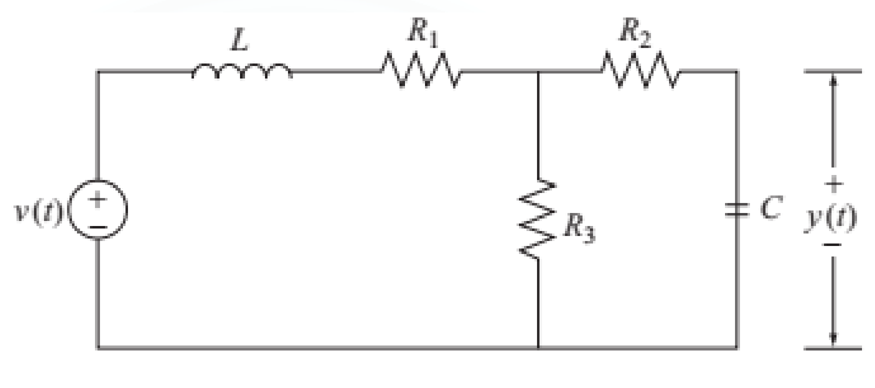
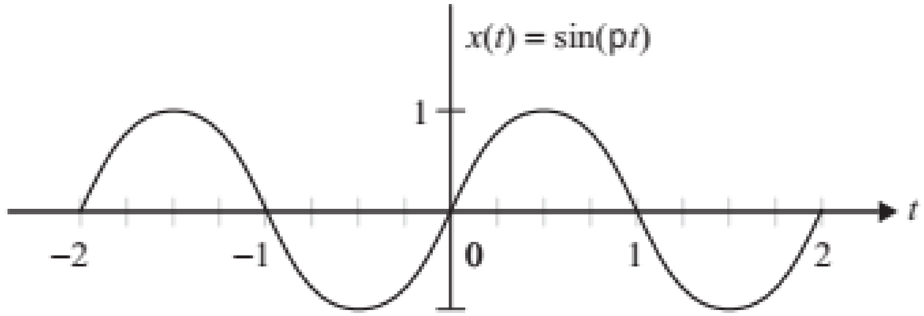
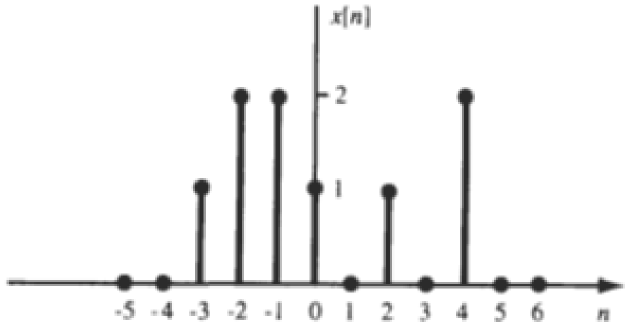
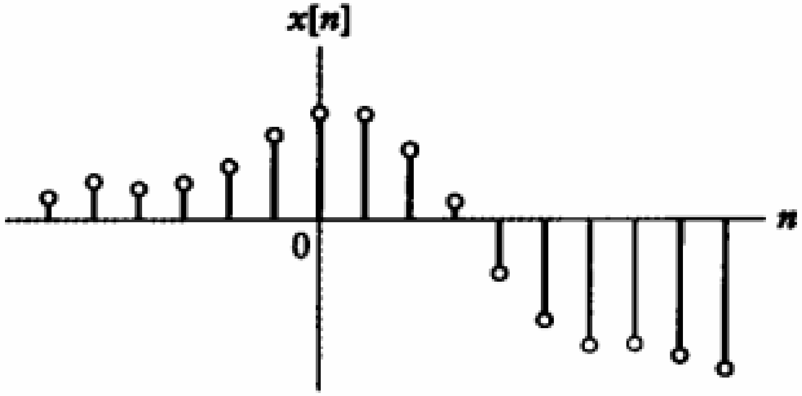
Sinyal Nedir:

- Signals are functions of time that represent the evolution of variables such as a furnace temperature, the speed of a car, a motor shaft position, or a voltage. Common examples of signals are human speech, temperature, pressure, and stock prices.
- Electrical signals, normally expressed in the form of voltage or current waveforms, are some of the easiest signals to generate and process. There are two types of signals: continuous-time signals and discrete-time signals.







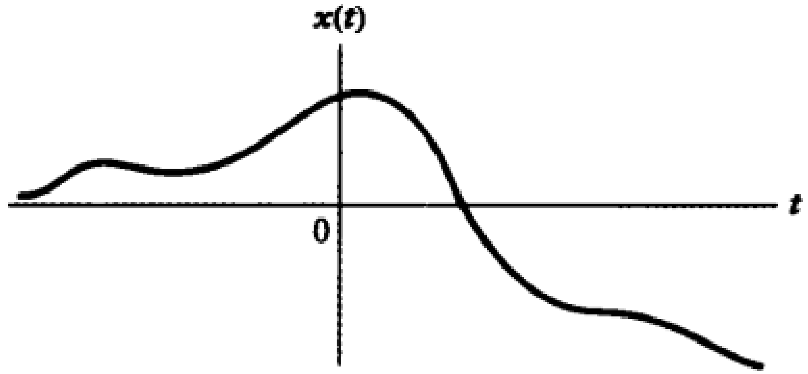


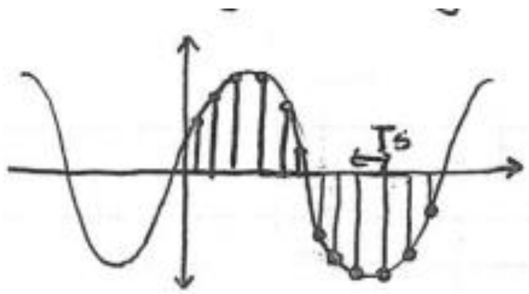
Sinyallerin sınıflandırılması:

- **Continuous-time signals:** If a signal is defined for all values of the independent variable t, it is called a continuous-time (CT) signal. Since these signals vary continuously with time t and have known magnitudes for all time instants, they are classified as CT signals.

In contrast, a *discrete-time signal* is defined only at discrete instants of time. Thus, the independent variable has discrete values only, which are usually uniformly spaced. A discrete-time signal is often derived from a continuous-time signal by *sampling* it at a uniform rate. Let T_s denote the sampling period and n denote an integer that may assume positive and negative values. Then sampling a continuous-time signal $x(t)$ at time $t = nT_s$ yields a sample with the value $x(nT_s)$. For convenience of presentation, we write

$$x[n] = x(nT_s), \qquad n = 0, \pm 1, \pm 2, \ldots \tag{1.1}$$





$x(t) = \sin t = x[n.T_s]$

$\Omega = \frac{2\Omega}{N} = \text{radyan}$

$n.T_s = x$

* $x[n]$ ayrık zamanlı sinyalinde n tamsayı dışında ifade edilemez.

A discrete-time signal $x[n]$ can be defined in two ways:

1. We can specify a rule for calculating the nth value of the sequence. For example,

$$x[n] = x_n = \begin{cases} \left(\frac{1}{2}\right)^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

or

$$\{x_n\} = \left\{1, \tfrac{1}{2}, \tfrac{1}{4}, \ldots, \left(\tfrac{1}{2}\right)^n, \ldots\right\}$$

2. We can also explicitly list the values of the sequence. For example, the sequence shown in Fig. 1-1(b) can be written as

$$\{x_n\} = \{\ldots, 0, 0, 1, 2, 2, \underset{\uparrow}{1}, 0, 1, 0, 2, 0, 0, \ldots\}$$

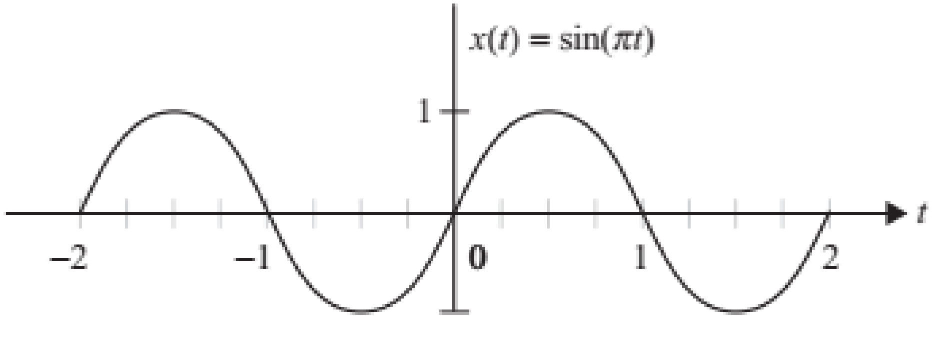
or

$$\{x_n\} = \{1, 2, 2, \underset{\uparrow}{1}, 0, 1, 0, 2\}$$

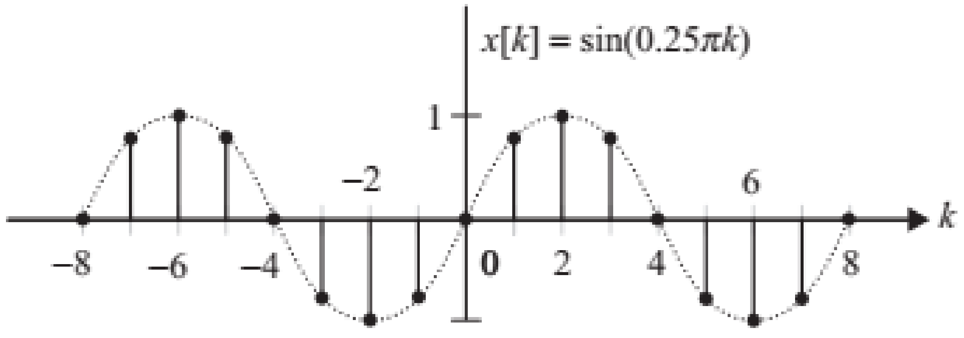
We use the arrow to denote the $n = 0$ term. We shall use the convention that if no arrow is indicated, then the first term corresponds to $n = 0$ and all the values of the sequence are zero for $n < 0$.

Example 1.1

Consider the CT signal $x(t) = \sin(\pi t)$ plotted in Fig. 1.3(a) as a function of time t. Discretize the signal using a sampling interval of $T = 0.25$ s, and sketch the waveform of the resulting DT sequence for the range $-8 \le k \le 8$.



(a)



(b)

For $k = 0, \pm 1, \pm 2, \ldots$, the DT signal $x[k]$ has the following values:

$x[-8] = x(-8T) = \sin(-2\pi) = 0,$

$x[-7] = x(-7T) = \sin(-1.75\pi) = \frac{1}{\sqrt{2}},$

$x[-6] = x(-6T) = \sin(-1.5\pi) = 1,$

$x[-5] = x(-5T) = \sin(-1.25\pi) = \frac{1}{\sqrt{2}},$

$x[-4] = x(-4T) = \sin(-\pi) = 0,$

$x[-3] = x(-3T) = \sin(-0.75\pi) = -\frac{1}{\sqrt{2}},$

$x[-2] = x(-2T) = \sin(-0.5\pi) = -1,$

$x[-1] = x(-T) = \sin(-0.25\pi) = -\frac{1}{\sqrt{2}},$

$x[1] = x(T) = \sin(0.25\pi) = \frac{1}{\sqrt{2}},$

$x[2] = x(2T) = \sin(0.5\pi) = 1,$

$x[3] = x(3T) = \sin(0.75\pi) = \frac{1}{\sqrt{2}},$

$x[4] = x(4T) = \sin(\pi) = 0,$

$x[5] = x(5T) = \sin(1.25\pi) = -\frac{1}{\sqrt{2}},$

$x[6] = x(6T) = \sin(1.5\pi) = -1,$

$x[7] = x(7T) = \sin(1.75\pi) = -\frac{1}{\sqrt{2}},$

$x[8] = x(8T) = \sin(2\pi) = 0,$

B. Analog and Digital Signals:

If a continuous-time signal $x(t)$ can take on any value in the continuous interval (a, b), where a may be $-\infty$ and b may be $+\infty$, then the continuous-time signal $x(t)$ is called an *analog* signal. If a discrete-time signal $x[n]$ can take on only a finite number of distinct values, then we call this signal a *digital* signal.

C. Real and Complex Signals:

A signal $x(t)$ is a *real* signal if its value is a real number, and a signal $x(t)$ is a *complex* signal if its value is a complex number. A general complex signal $x(t)$ is a function of the form

$$x(t) = x_1(t) + jx_2(t) \tag{1.1}$$

where $x_1(t)$ and $x_2(t)$ are real signals and $j = \sqrt{-1}$.

Note that in Eq. (*1.1*) t represents either a continuous or a discrete variable.

D. Deterministic and Random Signals:

Deterministic signals are those signals whose values are completely specified for any given time. Thus, a deterministic signal can be modeled by a known function of time t. *Random* signals are those signals that take random values at any given time and must be characterized statistically. Random signals will not be discussed in this text.

Odd and even signals

A CT signal $x_e(t)$ is said to be an even signal if

$$x_e(t) = x_e(-t).$$

Conversely, a CT signal $x_o(t)$ is said to be an odd signal if

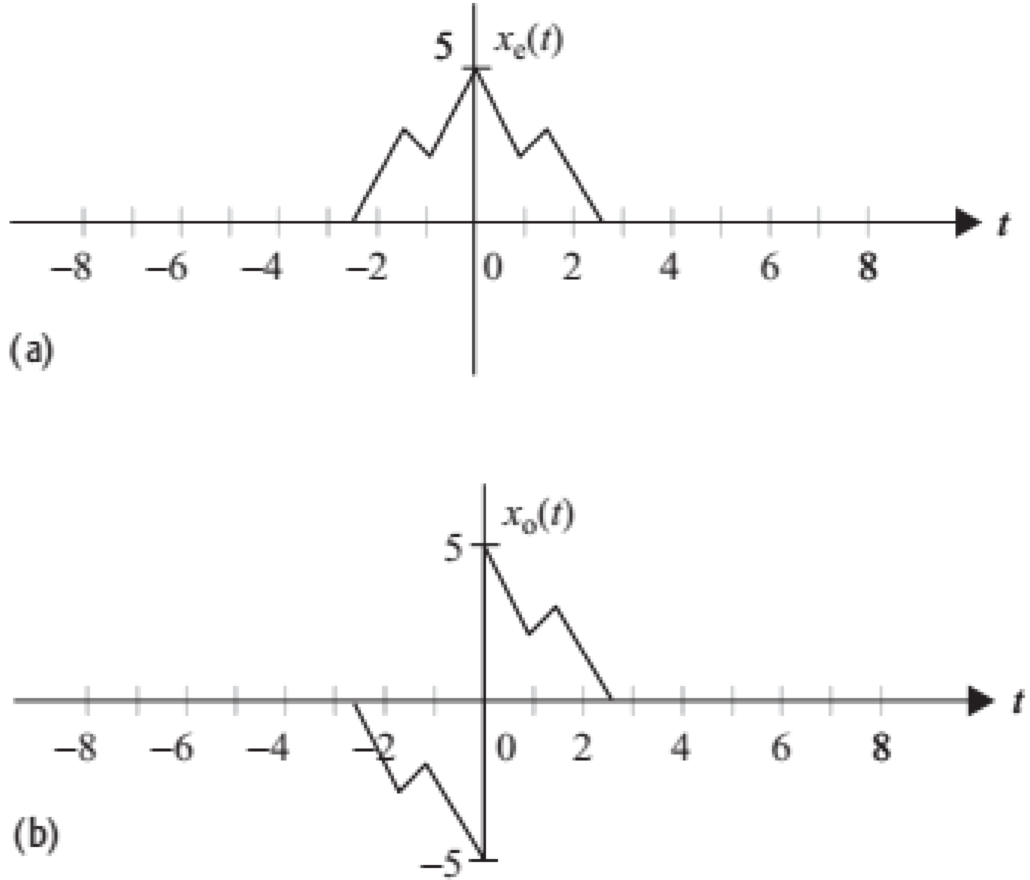
$$x_o(t) = -x_o(-t).$$

A DT signal $x_e[k]$ is said to be an even signal if

$$x_e[k] = x_e[-k].$$

Conversely, a DT signal $x_o[k]$ is said to be an odd signal if

$$x_o[k] = -x_o[-k].$$



Neither odd nor even signals can be expressed as a sum of even and odd signals as follows:

$$x(t) = x_e(t) + x_o(t),$$

where the even component $x_e(t)$ is given by

$$x_e(t) = \frac{1}{2}[x(t) + x(-t)],$$

while the odd component $x_o(t)$ is given by

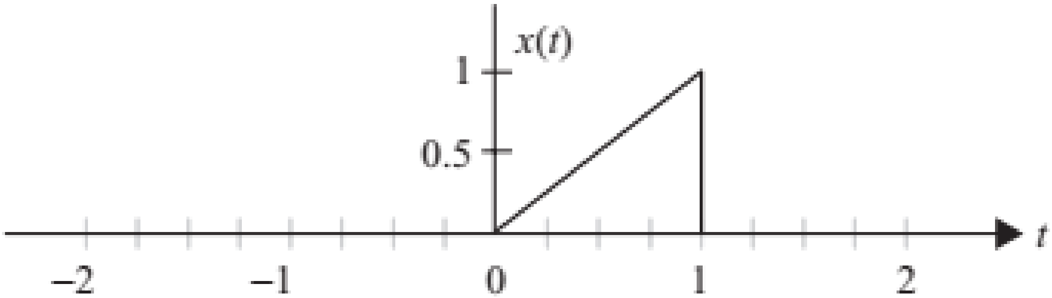
$$x_o(t) = \frac{1}{2}[x(t) - x(-t)].$$

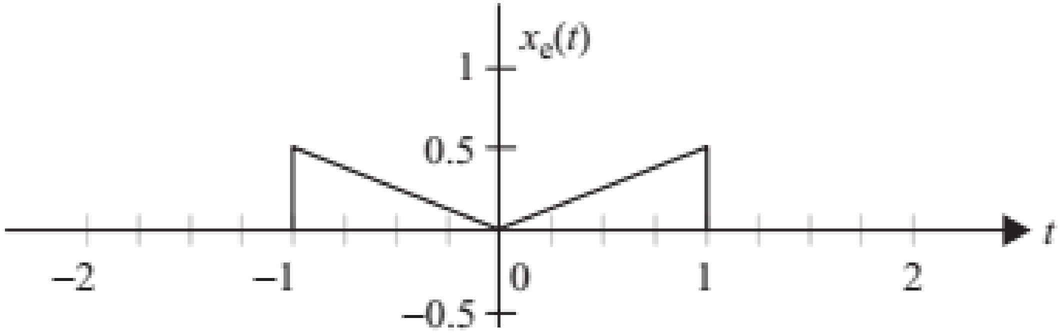
Example 1.9

Express the CT signal

$$x(t) = \begin{cases} t & 0 \le t < 1 \\ 0 & \text{elsewhere} \end{cases}$$

as a combination of an even signal and an odd signal.





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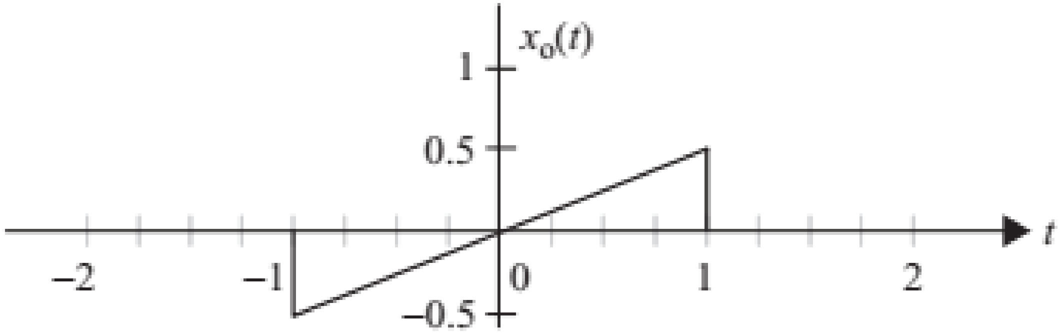


(c)

EXAMPLE 1.2 ANOTHER EXAMPLE OF EVEN AND ODD SIGNALS Find the even and odd components of the signal

$$x(t) = e^{-2t} \cos t.$$

Solution: Replacing t with $-t$ in the expression for $x(t)$ yields

$$\begin{aligned} x(-t) &= e^{2t} \cos(-t) \\ &= e^{2t} \cos t. \end{aligned}$$

$$\begin{aligned} x_e(t) &= \frac{1}{2}(e^{-2t} \cos t + e^{2t} \cos t) \\ &= \cosh(2t) \cos t \end{aligned} \qquad \begin{aligned} x_o(t) &= \frac{1}{2}(e^{-2t} \cos t - e^{2t} \cos t) \\ &= -\sinh(2t) \cos t, \end{aligned}$$

$x(t) = 1 + t + 3t^2 + 5t^3 + 9t^4$ $\rightarrow$ Even: $1 + 3t^2 + 9t^4$ Odd: $t + 5t^3$

$x(t) = (1 + t^3) \cos^3(10t)$ $\rightarrow$ Even: $\cos^3(10t)$ Odd: $t^3 \cos^3(10t)$

In the case of a complex-valued signal, we may speak of conjugate symmetry. A complex-valued signal $x(t)$ is said to be *conjugate symmetric* if

$$x(-t) = x^*(t), \qquad (1.6)$$

where the asterisk denotes complex conjugation. Let

$$x(t) = a(t) + jb(t),$$

where $a(t)$ is the real part of $x(t)$, $b(t)$ is the imaginary part, and $j = \sqrt{-1}$. Then the complex conjugate of $x(t)$ is

$$x^*(t) = a(t) - jb(t).$$

Substituting $x(t)$ and $x^*(t)$ into Eq. (1.6) yields

$$a(-t) + jb(-t) = a(t) - jb(t).$$

▶ **Problem 1.2** The signals $x_1(t)$ and $x_2(t)$ shown in Figs. 1.13(a) and (b) constitute the real and imaginary parts, respectively, of a complex-valued signal $x(t)$. What form of symmetry does $x(t)$ have?

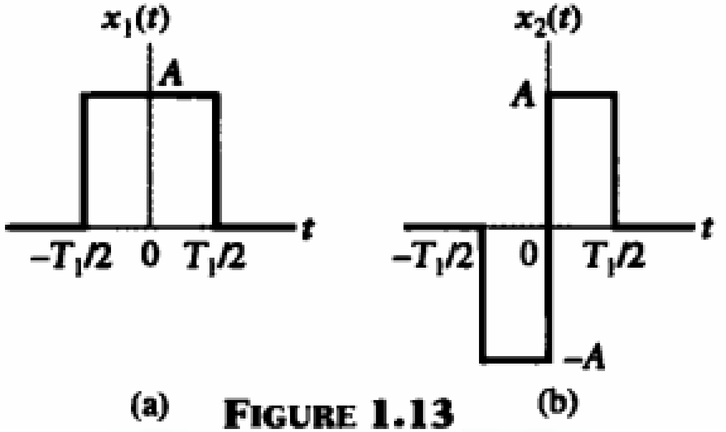


FIGURE 1.13

The signal $x(t)$ is conjugate symmetric.

Combinations of even and odd CT signals

Consider $g_e(t)$ and $h_e(t)$ as two CT even signals and $g_o(t)$ and $h_o(t)$ as two CT odd signals. The following properties may be used to classify different combinations of these four signals into the even and odd categories.

(i) Multiplication of a CT even signal with a CT odd signal results in a CT odd signal. The CT signal $x(t) = g_e(t) \times g_o(t)$ is therefore an odd signal.

(ii) Multiplication of a CT odd signal with another CT odd signal results in a CT even signal. The CT signal $h(t) = g_o(t) \times h_o(t)$ is therefore an even signal.

(iii) Multiplication of two CT even signals results in another CT even signal. The CT signal $z(t) = g_e(t) \times h_e(t)$ is therefore an even signal.

(iv) Due to its antisymmetry property, a CT odd signal is always zero at $t = 0$. Therefore, $g_o(0) = h_o(0) = 0$.

(v) Integration of a CT odd signal within the limits $[-T, T]$ results in a zero value, i.e.

$$\int_{-T}^{T} g_o(t)\mathrm{d}t = \int_{-T}^{T} h_o(t)\mathrm{d}t = 0. \qquad \int_{-T}^{T} g_e(t)\mathrm{d}t = 2\int_{0}^{T} g_e(t)\mathrm{d}t.$$

Combinations of even and odd DT signals

Properties (i)–(vi) for CT signals can be extended to DT sequences. Consider $g_e[k]$ and $h_e[k]$ as even sequences and $g_o[k]$ and $h_o[k]$ are as odd sequences. For the four DT signals, the following properties hold true.

(i) Multiplication of an even sequence with an odd sequence results in an odd sequence. The DT sequence $x[k] = g_e[k] \times g_o[k]$, for example, is an odd sequence.

(ii) Multiplication of two odd sequences results in an even sequence. The DT sequence $h[k] = g_o[k] \times h_o[k]$, for example, is an even sequence.

(iii) Multiplication of two even sequences results in an even sequence. The DT sequence $z[k] = g_e[k] \times h_e[k]$, for example, is an even sequence.

(iv) Due to its antisymmetry property, a DT odd sequence is always zero at $k = 0$. Therefore, $g_o[0] = h_o[0] = 0$.

(v) Adding the samples of a DT odd sequence $g_o[k]$ within the range $[-M, M]$ is 0, i.e.

$$\sum_{k=-M}^{M} g_o[k] = 0 = \sum_{k=-M}^{M} h_o[k].$$

(vi) Adding the samples of a DT even sequence $g_e[k]$ within the range $[-M, M]$ simplifies to

$$\sum_{k=-M}^{M} g_e[k] = g_e[0] + 2\sum_{k=1}^{M} g_e[k].$$

Periodic and aperiodic signals

A *periodic signal* $x(t)$ is a function of time that satisfies the condition

$$x(t) = x(t + T) \qquad \text{for all } t, \tag{1.7}$$

where T is a positive constant. Clearly, if this condition is satisfied for $T = T_0$, say, then it is also satisfied for $T = 2T_0, 3T_0, 4T_0, \ldots$. The smallest value of T that satisfies Eq. (1.7) is called the *fundamental period* of $x(t)$. Accordingly, the fundamental period T defines the duration of one complete cycle of $x(t)$. The reciprocal of the fundamental period T is called the *fundamental frequency* of the periodic signal $x(t)$; it describes how frequently the periodic signal $x(t)$ repeats itself. We thus formally write

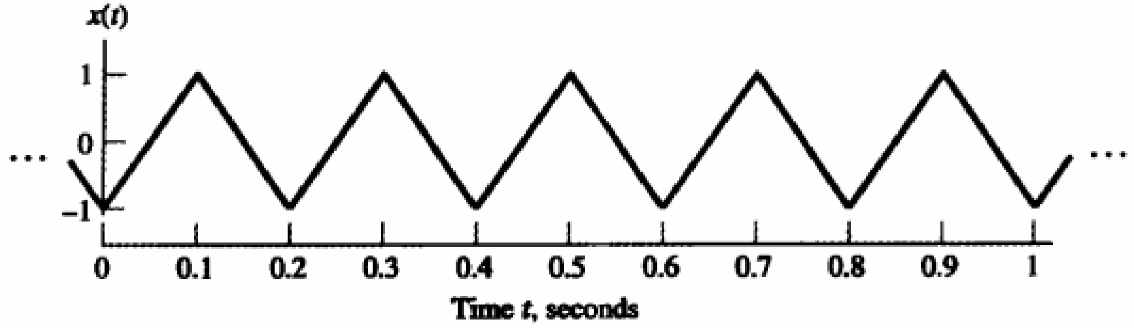
$$f = \frac{1}{T}. \tag{1.8}$$

The frequency f is measured in hertz (Hz), or cycles per second. The *angular frequency*, measured in radians per second, is defined by

$$\omega = 2\pi f = \frac{2\pi}{T}, \tag{1.9}$$

► **Problem 1.3** Figure 1.15 shows a triangular wave. What is the fundamental frequency of this wave? Express the fundamental frequency in units of Hz and rad/s.

Answer: 5 Hz, or 10π rad/s.



The classification of signals into periodic and nonperiodic signals presented thus far applies to continuous-time signals. We next consider the case of discrete-time signals. A discrete-time signal $x[n]$ is said to be periodic if

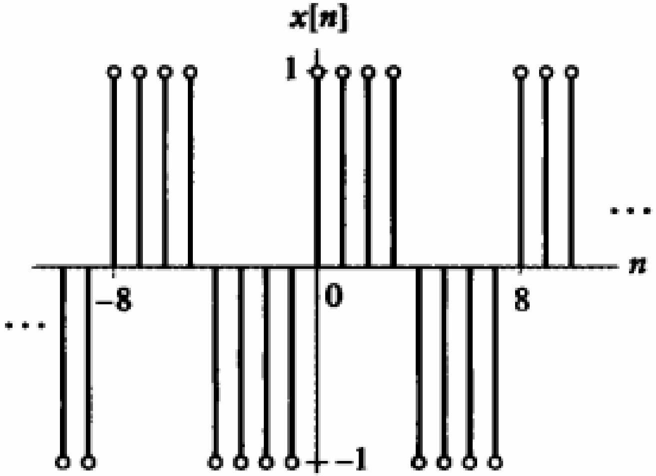
$$x[n] = x[n + N] \qquad \text{for integer } n, \tag{1.10}$$

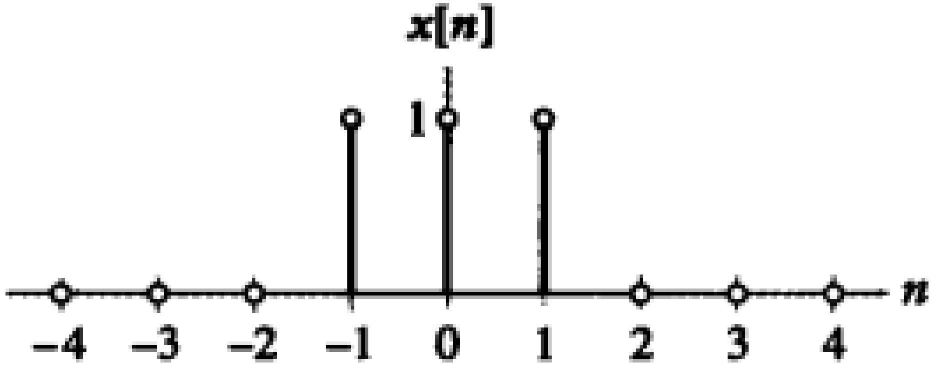
where N is a positive integer. The smallest integer N for which Eq. (1.10) is satisfied is called the fundamental period of the discrete-time signal $x[n]$. The fundamental angular frequency or, simply, fundamental frequency of $x[n]$ is defined by

$$\Omega = \frac{2\pi}{N}, \tag{1.11}$$

which is measured in radians.

The differences between the defining equations (1.7) and (1.10) should be carefully noted. Equation (1.7) applies to a periodic continuous-time signal whose fundamental period T has any positive value. Equation (1.10) applies to a periodic discrete-time signal whose fundamental period N can assume only a positive integer value.





Proposition 1.1 *An arbitrary DT sinusoidal sequence $x[k] = A\sin(\Omega_0 k + \theta)$ is periodic iff $\Omega_0/2\pi$ is a rational number.*

The term *rational number* used in Proposition 1.1 is defined as a fraction of two integers. Given that the DT sinusoidal sequence $x[k] = A\sin(\Omega_0 k + \theta)$ is periodic, its fundamental period is evaluated from the relationship

$$\frac{\Omega_0}{2\pi} = \frac{m}{K_0} \tag{1.7}$$

as

$$K_0 = \frac{2\pi}{\Omega_0} m. \tag{1.8}$$

Proposition 1.1 can be extended to include DT complex exponential signals. Collectively, we state the following.

(1) The fundamental period of a sinusoidal signal that satisfies Proposition 1.1 is calculated from Eq. (1.8) with m set to the smallest integer that results in an integer value for K_0.
(2) A complex exponential $x[k] = A\exp[j(\Omega_0 k + \theta)]$ must also satisfy Proposition 1.1 to be periodic. The fundamental period of a complex exponential is also given by Eq. (1.8).

Example 1.4

Determine if the sinusoidal DT sequences (i)–(iv) are periodic:

(i) $f[k] = \sin(\pi k/12 + \pi/4)$;
(ii) $g[k] = \cos(3\pi k/10 + \theta)$;
(iii) $h[k] = \cos(0.5k + \phi)$;
(iv) $p[k] = e^{j(7\pi k/8+\theta)}$.

Solution

(i) The value of Ω_0 in $f[k]$ is $\pi/12$. Since $\Omega_0/2\pi = 1/24$ is a rational number, the DT sequence $f[k]$ is periodic. Using Eq. (1.8), the fundamental period of $f[k]$ is given by

$$K_0 = \frac{2\pi}{\Omega_0}m = 24m.$$

Setting $m = 1$ yields the fundamental period $K_0 = 24$.

To demonstrate that $f[k]$ is indeed a periodic signal, consider the following:

$$f[k + K_0] = \sin(\pi[k + K_0]/12 + \pi/4).$$

Substituting $K_0 = 24$ in the above equation, we obtain

$$\begin{aligned} f[k + K_0] &= \sin(\pi[k + K_0]/12 + \pi/4) = \sin(\pi k + 2\pi + \pi/4) \\ &= \sin(\pi k/12 + \pi/4) = f[k]. \end{aligned}$$

(ii) The value of Ω_0 in $g[k]$ is $3\pi/10$. Since $\Omega_0/2\pi = 3/20$ is a rational number, the DT sequence $g[k]$ is periodic. Using Eq. (1.8), the fundamental period of $g[k]$ is given by

$$K_0 = \frac{2\pi}{\Omega_0}m = \frac{20m}{3}.$$

Setting $m = 3$ yields the fundamental period $K_0 = 20$.

(iii) The value of Ω_0 in $h[k]$ is 0.5. Since $\Omega_0/2\pi = 1/4\pi$ is not a rational number, the DT sequence $h[k]$ is not periodic.

(iv) The value of Ω_0 in $p[k]$ is $7\pi/8$. Since $\Omega_0/2\pi = 7/16$ is a rational number, the DT sequence $p[k]$ is periodic. Using Eq. (1.8), the fundamental period of $p[k]$ is given by

$$K_0 = \frac{2\pi}{\Omega_0}m = \frac{16m}{7}.$$

Setting $m = 7$ yields the fundamental period $K_0 = 16$.

▶ **Problem 1.5** For each of the following signals, determine whether it is periodic, and if it is, find the fundamental period:

(a) $x(t) = \cos^2(2\pi t)$
(b) $x(t) = \sin^3(2t)$
(c) $x(t) = e^{-2t}\cos(2\pi t)$
(d) $x[n] = (-1)^n$
(e) $x[n] = (-1)^{n^2}$
(f) $x[n] = \cos(2n)$
(g) $x[n] = \cos(2\pi n)$

Answers:

(a) Periodic, with a fundamental period of 0.5 s
(b) Periodic, with a fundamental period of $(1/\pi)$ s
(c) Nonperiodic
(d) Periodic, with a fundamental period of 2 samples
(e) Periodic, with a fundamental period of 2 samples
(f) Nonperiodic
(g) Periodic, with a fundamental period of 1 sample

Basic Operations on Signals

Operations Performed on the Independent Variable

The *time-scaling* operation compresses or expands the input signal in the time domain. A CT signal $x(t)$ scaled by a factor c in the time domain is denoted by $x(ct)$. If $c > 1$, the signal is compressed by a factor of c. On the other hand, if $0 < c < 1$ the signal is expanded. We illustrate the concept of time scaling of CT signals with the help of a few examples.

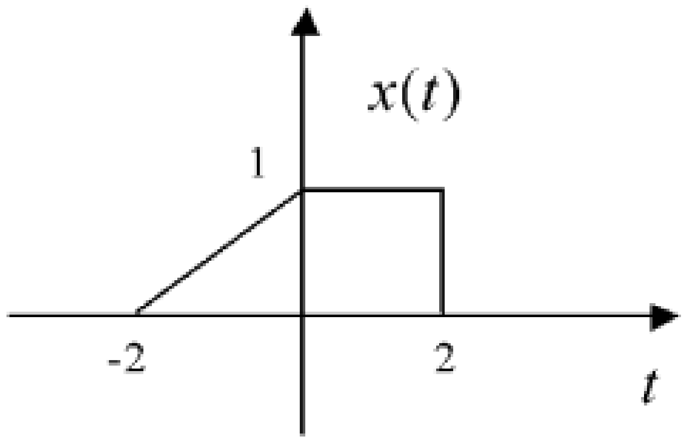


FIGURE 1.4 Graph of continuous time signal $x(t)$.

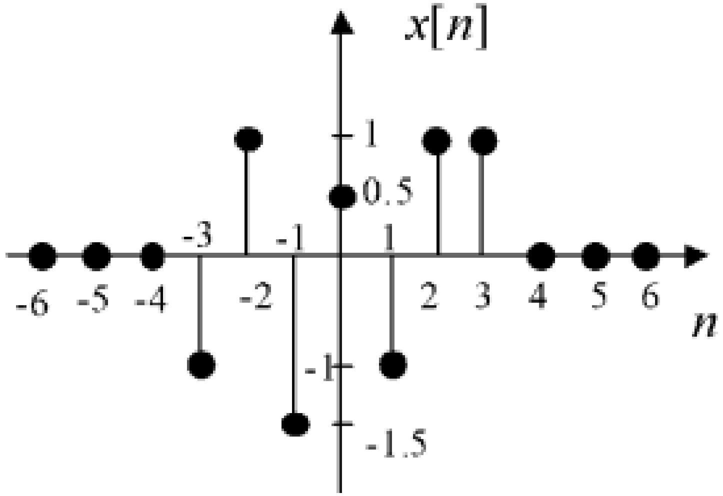


FIGURE 1.5 Graph of discrete-time signal $x[n]$.

Example 1.3: Case $\alpha = \frac{1}{2}$ shown in Figure 1.6.

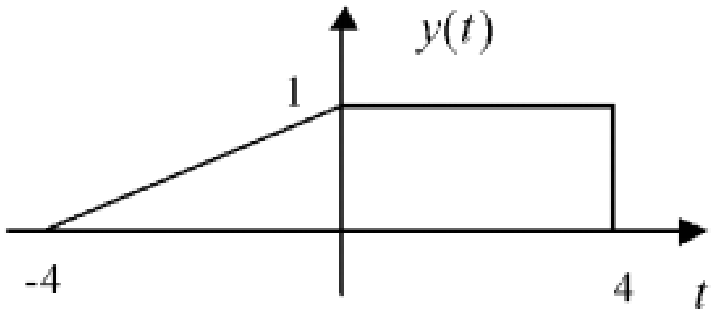


FIGURE 1.6 Graph of expanded signal $y(t) = x(0.5t)$.

Case $0 < \alpha < 1$: The signal $x(t)$ is *slowed down* or *expanded* in time. Think of a tape recording played back at a slower speed than the nominal speed.

$y[n] = x[2n]$

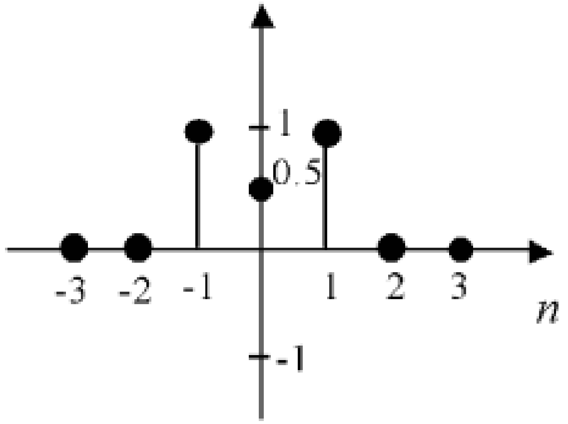


FIGURE 1.8 Graph of compressed signal $y[n] = x[2n]$.

Example 1.4: Case $\alpha = 2$ shown in Figure 1.7.

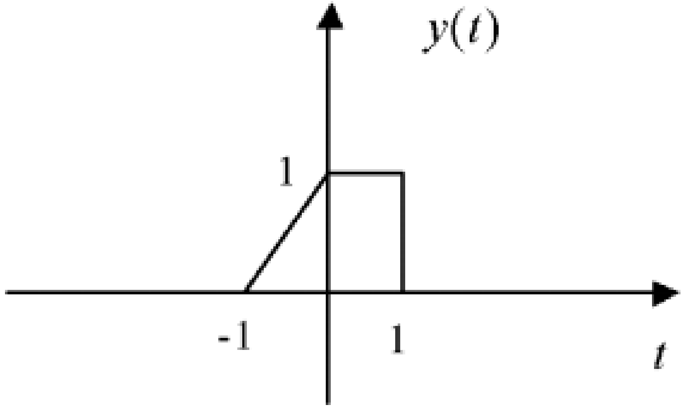


FIGURE 1.7 Graph of compressed signal $y(t) = x(2t)$.

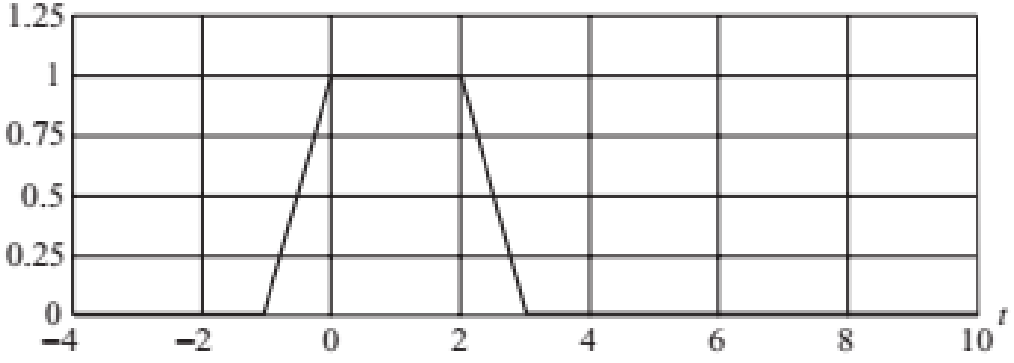
Case $\alpha > 1$: The signal $x(t)$ is *sped up* or *compressed* in time. Think of a tape recording played back at twice the nominal speed.

Example 1.16

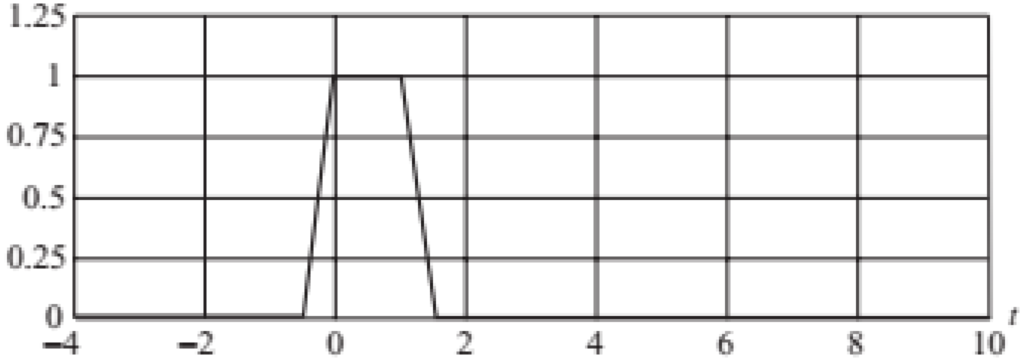
Consider a CT signal $x(t)$ defined as follows:

$$x(t) = \begin{cases} t+1 & -1 \le t \le 0 \\ 1 & 0 \le t \le 2 \\ -t+3 & 2 \le t \le 3 \\ 0 & \text{elsewhere}, \end{cases} \tag{1.53}$$

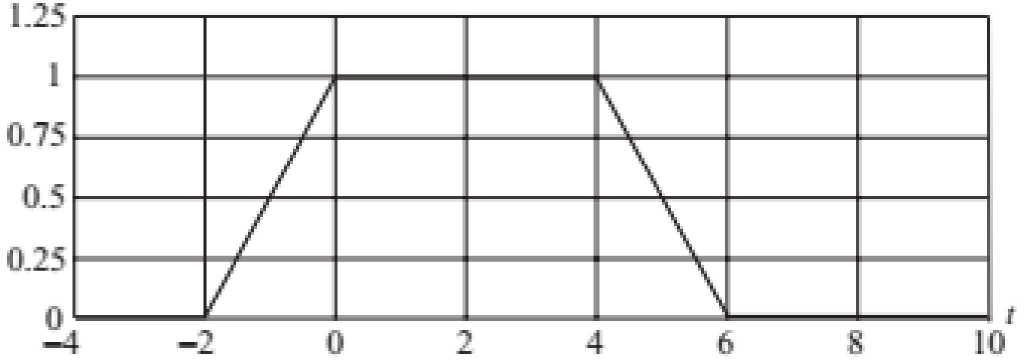
as plotted in Fig. 1.25(a). Determine the expressions for the time-scaled signals $x(2t)$ and $x(t/2)$. Sketch the two signals.



(a)



(b)



(c)

Time Reversal

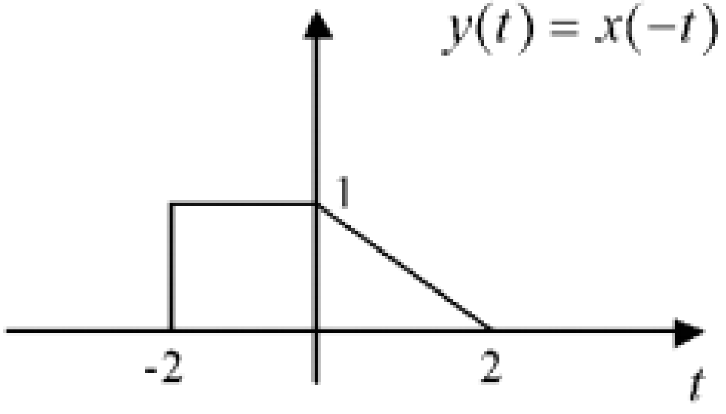


FIGURE 1.9 Graph of time-reversed signal $y(t) = x(-t)$.

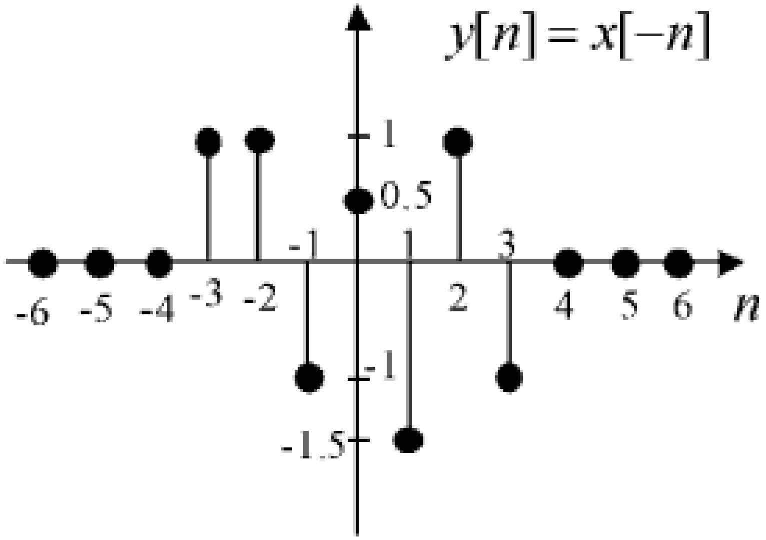


FIGURE 1.10 Graph of time-reversed signal $y[n] = x[-n]$.

Time Shift

A time shift delays or advances the signal in time by a continuous-time interval $T \in \mathbb{R}$:

$$y(t) = x(t + T). \tag{1.3}$$

For T positive, the signal is advanced; that is, it starts at time $t = -4$, which is before the time it originally started at, $t = -2$, as shown in Figure 1.11. For T negative, the signal is delayed, as shown in Figure 1.12.

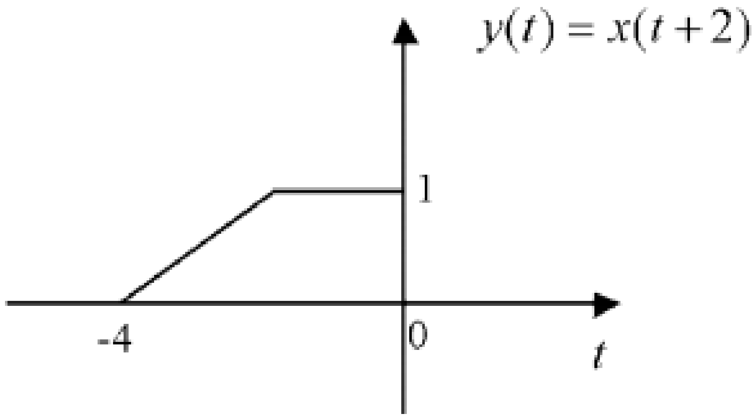


FIGURE 1.11 Graph of time-advanced signal $y(t) = x(t + 2)$.

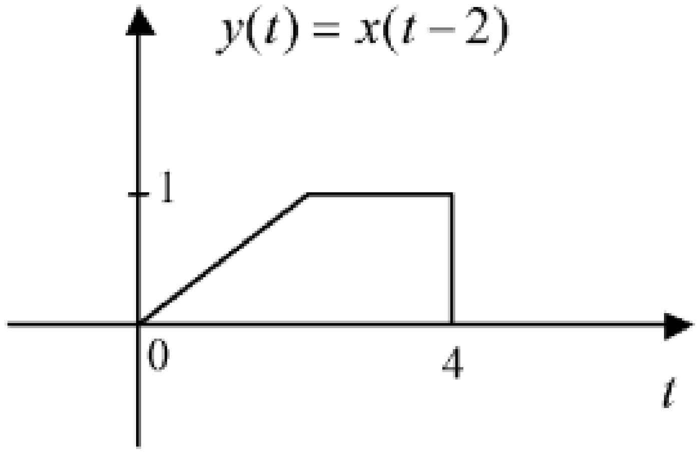


FIGURE 1.12 Graph of time-delayed signal $y(t) = x(t - 2)$.

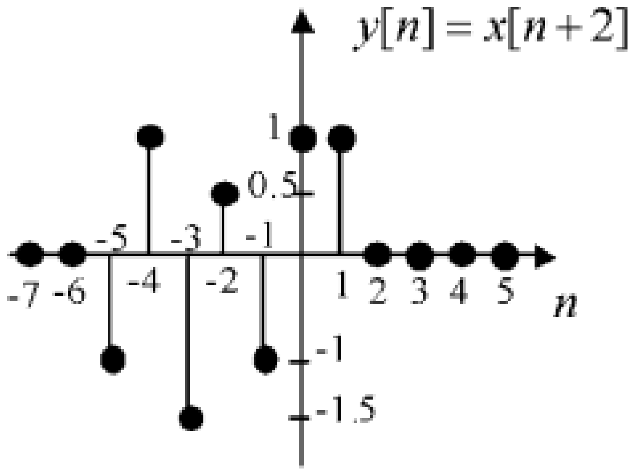


FIGURE 1.13 Graph of time-advanced signal $y[n] = x[n + 2]$.

Amplitude scaling. Let $x(t)$ denote a continuous-time signal. Then the signal $y(t)$ resulting from amplitude scaling applied to $x(t)$ is defined by

$$y(t) = cx(t), \tag{1.21}$$

where c is the scaling factor. According to Eq. (1.21), the value of $y(t)$ is obtained by multiplying the corresponding value of $x(t)$ by the scalar c for each instant of time t. A physical example of a device that performs amplitude scaling is an electronic *amplifier*. A resistor also performs amplitude scaling when $x(t)$ is a current, c is the resistance of the resistor, and $y(t)$ is the output voltage.

In a manner similar to Eq. (1.21), for discrete-time signals, we write

$$y[n] = cx[n].$$

Addition. Let $x_1(t)$ and $x_2(t)$ denote a pair of continuous-time signals. Then the signal $y(t)$ obtained by the addition of $x_1(t)$ and $x_2(t)$ is defined by

$$y(t) = x_1(t) + x_2(t). \tag{1.22}$$

A physical example of a device that adds signals is an audio *mixer*, which combines music and voice signals.

In a manner similar to Eq. (1.22), for discrete-time signals, we write

$$y[n] = x_1[n] + x_2[n].$$

Multiplication. Let $x_1(t)$ and $x_2(t)$ denote a pair of continuous-time signals. Then the signal $y(t)$ resulting from the multiplication of $x_1(t)$ by $x_2(t)$ is defined by

$$y(t) = x_1(t)x_2(t). \tag{1.23}$$

That is, for each prescribed time t, the value of $y(t)$ is given by the product of the corresponding values of $x_1(t)$ and $x_2(t)$. A physical example of $y(t)$ is an *AM radio signal*, in which $x_1(t)$ consists of an audio signal plus a dc component and $x_2(t)$ consists of a sinusoidal signal called a carrier wave.

In a manner similar to Eq. (1.23), for discrete-time signals, we write

$$y[n] = x_1[n]x_2[n].$$

Differentiation. Let $x(t)$ denote a continuous-time signal. Then the derivative of $x(t)$ with respect to time is defined by

$$y(t) = \frac{d}{dt}x(t). \tag{1.24}$$

$$v(t) = L\frac{d}{dt}i(t). \tag{1.25}$$

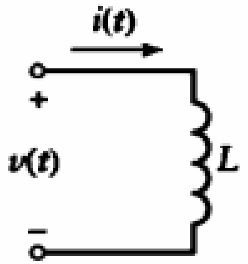
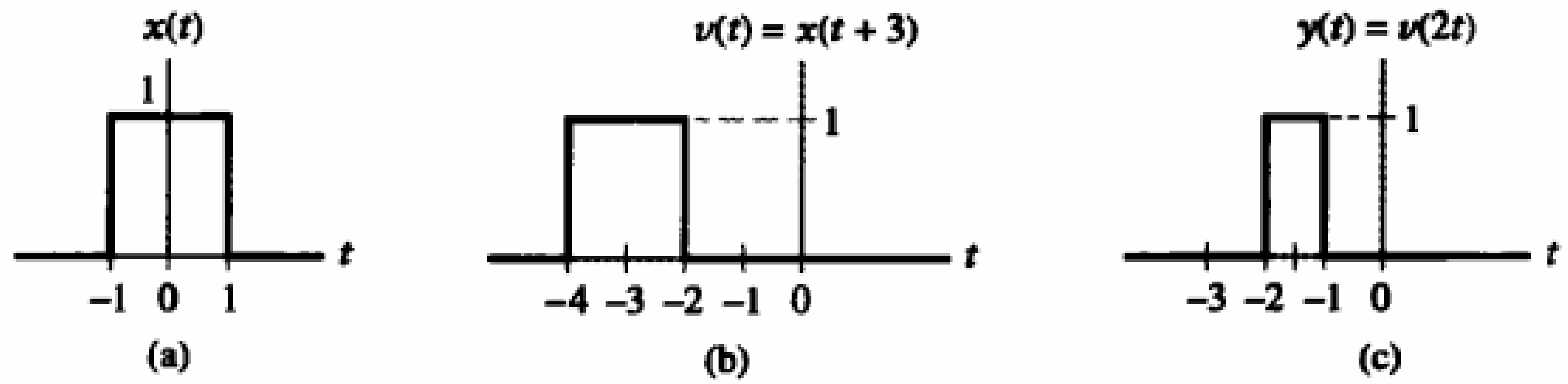


FIGURE 1.18 Inductor with current $i(t)$, inducing voltage $v(t)$ across its terminals.

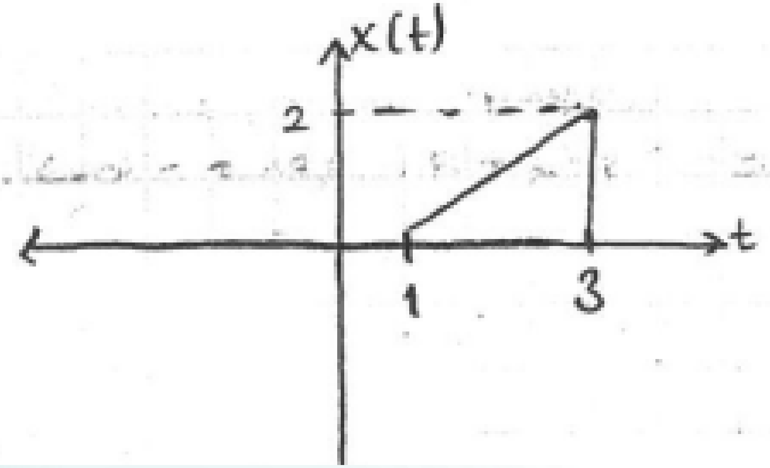
Precedence Rule for Time Shifting and Time Scaling

KURAL: Önce ekseni kaydır
sonra değişkeni ölçeklendir
sonra genlik varsa çarp.

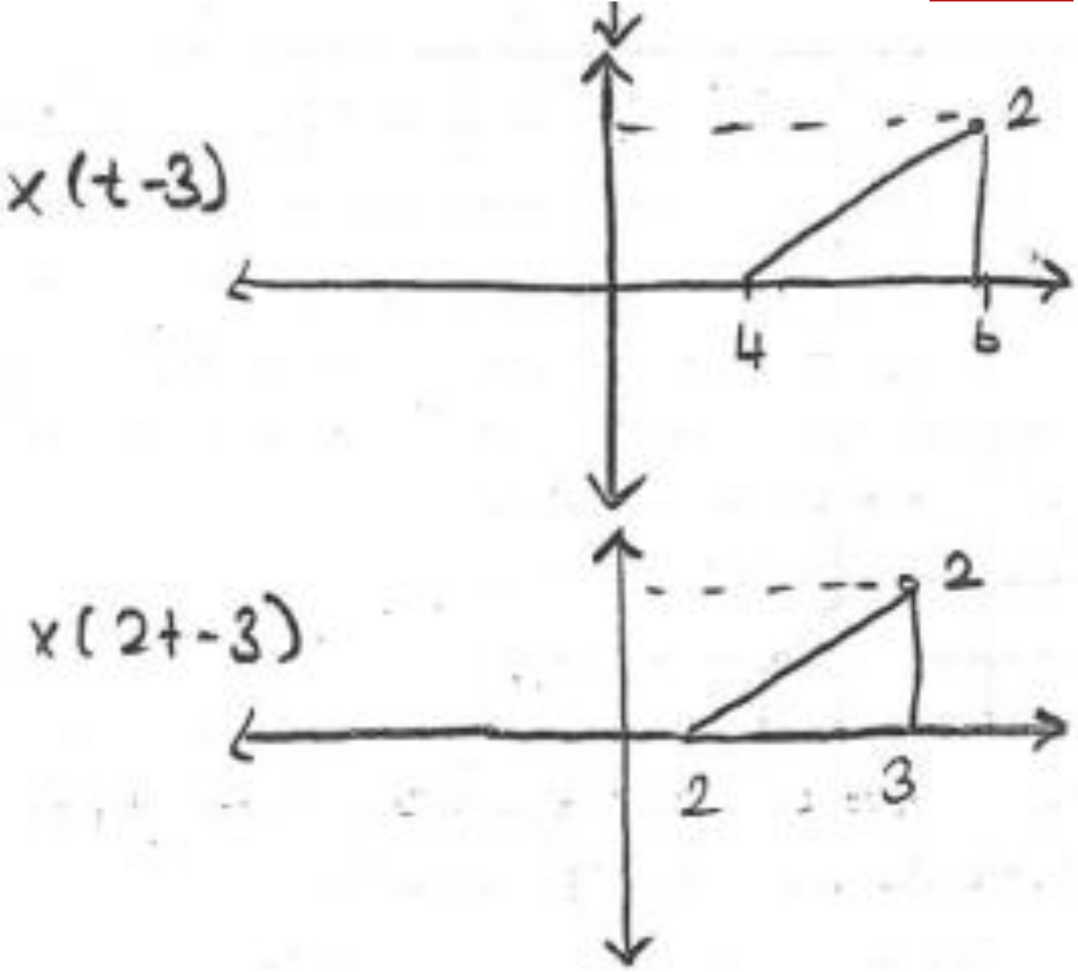
Example 1.5 Precedence Rule for Continuous-Time Signal Consider the rectangular pulse $x(t)$ of unit amplitude and a duration of 2 time units, depicted in Fig. 1.24(a). Find $y(t) = x(2t + 3)$.



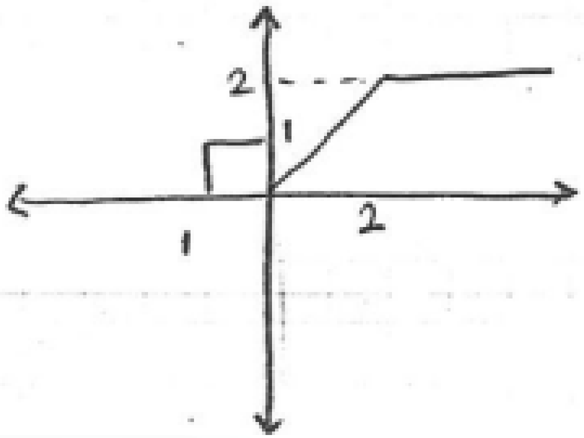
Örnek:



$x(t) \longrightarrow x(2t-3)$

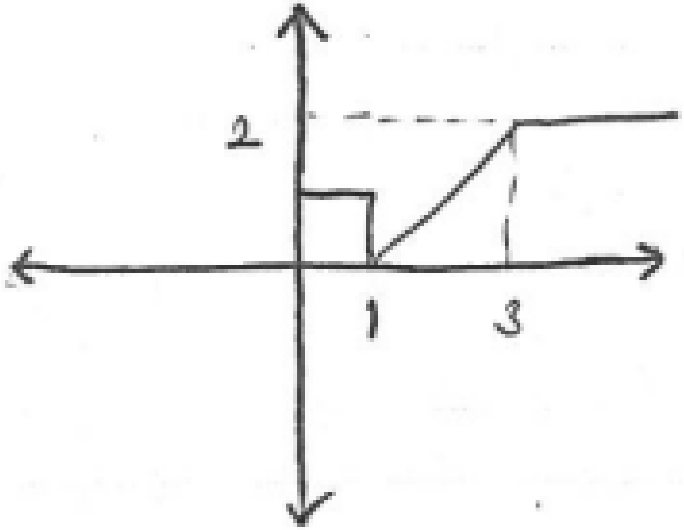


Örnek:

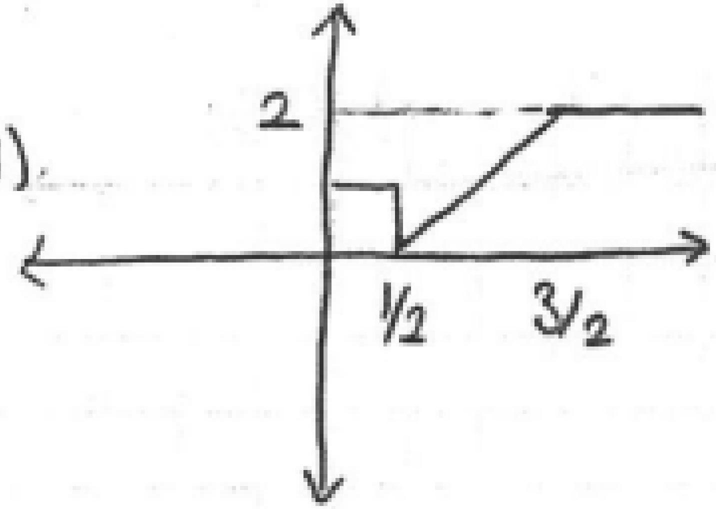


$x(t) \longrightarrow x(2t-1)$

$x(t-1)$



$x(2t-1)$



▶ **Problem 1.14** A triangular pulse signal $x(t)$ is depicted in Fig. 1.26. Sketch each of the following signals derived from $x(t)$:

(a) $x(3t)$

(b) $x(3t + 2)$

(c) $x(-2t - 1)$

(d) $x(2(t + 2))$

(e) $x(2(t - 2))$

(f) $x(3t) + x(3t + 2)$

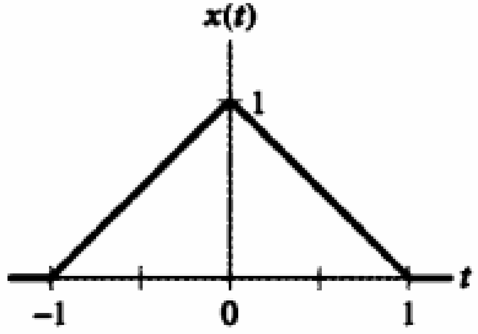
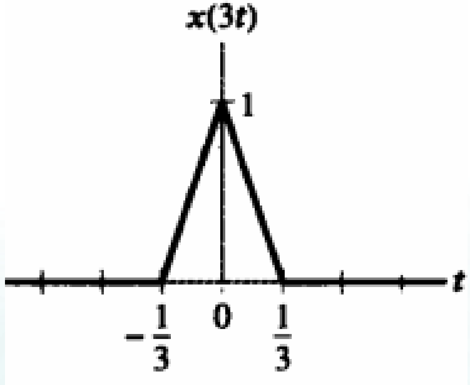
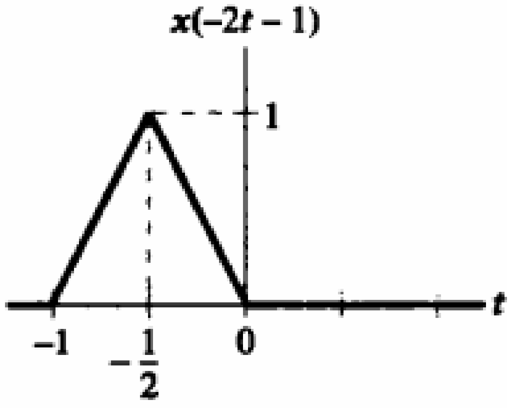


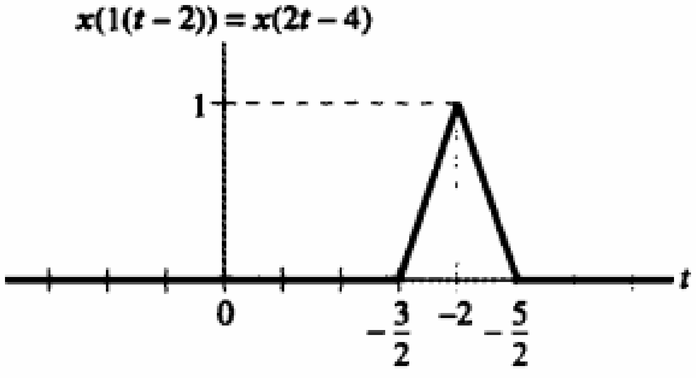
FIGURE 1.26 Triangular pulse for Problem 1.14.



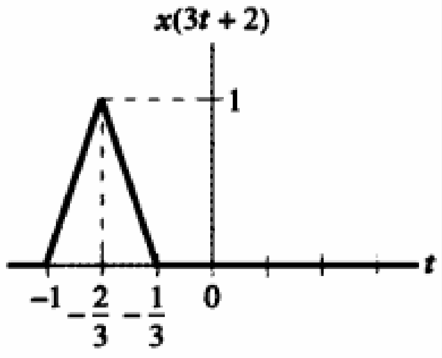
(a)



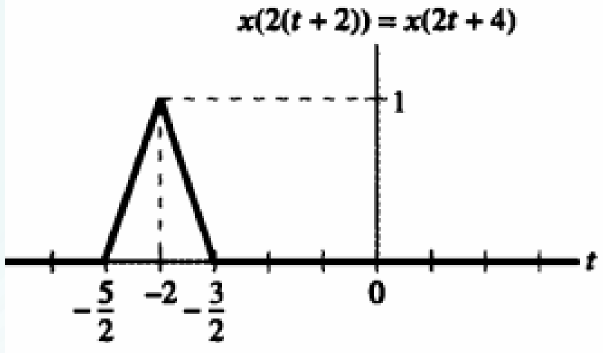
(c)



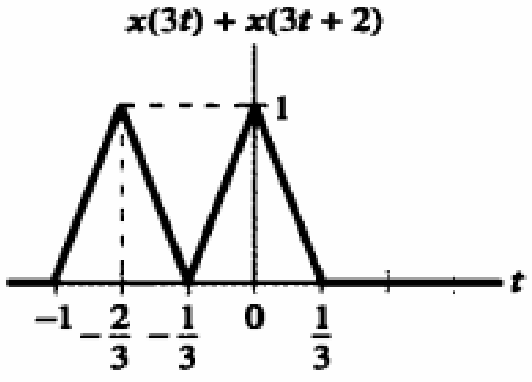
(e)



(b)



(d)



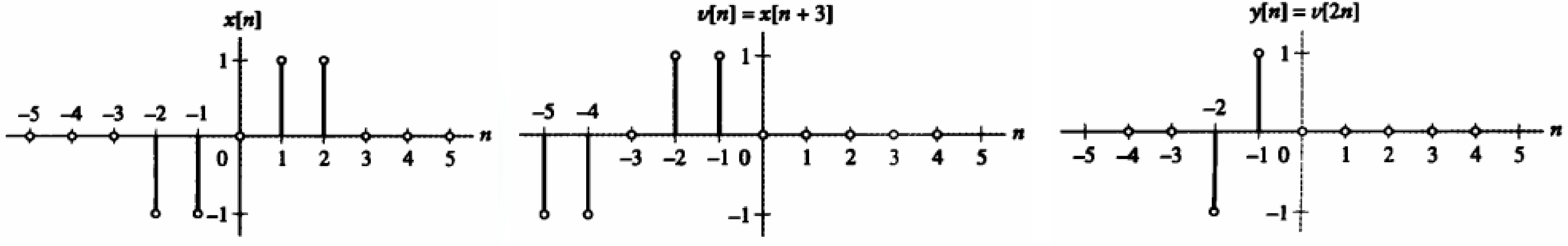
(f)

EXAMPLE 1.6 PRECEDENCE RULE FOR DISCRETE-TIME SIGNAL A discrete-time signal is defined by

$$x[n] = \begin{cases} 1, & n = 1, 2 \\ -1, & n = -1, -2 \\ 0, & n = 0 \text{ and } |n| > 2 \end{cases}.$$

Find $y[n] = x[2n + 3]$.

Solution:



(a) (b) (c)

FINITE-ENERGY AND FINITE-POWER SIGNALS

The instantaneous power dissipated in a resistor of resistance R is simply the product of the voltage across and the current through the resistor:

$$p(t) = v(t)i(t) = \frac{v^2(t)}{R},$$

and the *total energy* dissipated during a time interval $[t_1, t_2]$ is obtained by integrating the power

$$E_{[t_1,t_2]} = \int_{t_1}^{t_2} p(t)dt = \int_{t_1}^{t_2} \frac{v^2(t)}{R} dt.$$

The *average power* dissipated over that interval is the total energy divided by the time interval:

$$P_{[t_1,t_2]} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \frac{v^2(t)}{R} dt.$$

Analogously, the total energy and average power over $[t_1, t_2]$ of an arbitrary integrable continuous-time signal $x(t)$ are defined as though the signal were a voltage across a one-ohm resistor:

$$E_{[t_1,t_2]} := \int_{t_1}^{t_2} |x(t)|^2 \, dt,$$

$$P_{[t_1,t_2]} := \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |x(t)|^2 \, dt.$$

The total energy and total average power of a signal defined over $t \in \mathbb{R}$ are defined as

$$E_\infty := \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |x(t)|^2 \, dt,$$

$$P_\infty := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt.$$

The total energy and average power over $[n_1, n_2]$ of an arbitrary discrete-time signal $x[n]$ are defined as

$$E_{[n_1,n_2]} := \sum_{n=n_1}^{n_2} |x[n]|^2,$$

$$P_{[n_1,n_2]} := \frac{1}{n_2 - n_1 + 1} \sum_{n=n_1}^{n_2} |x[n]|^2.$$

Notice that $n_2 - n_1 + 1$ is the number of points in the signal over the interval $[n_1, n_2]$. The total energy and total average power of signal $x[n]$ defined over $n \in \mathbb{Z}$ are defined as

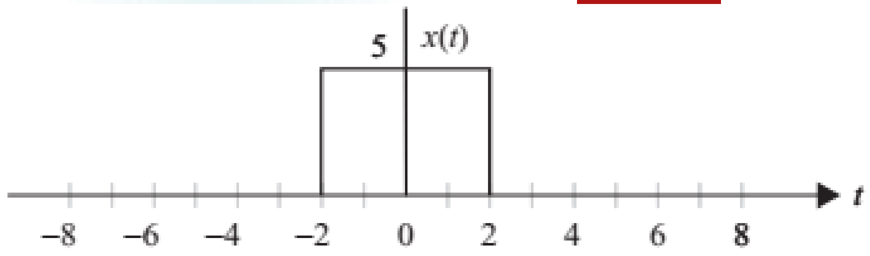
$$E_\infty := \lim_{N\to\infty} \sum_{n=-N}^{N} |x[n]|^2 = \sum_{n=-\infty}^{\infty} |x[n]|^2,$$

$$P_\infty := \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2.$$

A signal $x(t)$, or $x[k]$, is called an *energy signal* if the total energy E_x has a non-zero finite value, i.e. $0 < E_x < \infty$. On the other hand, a signal is called a *power signal* if it has non-zero finite power, i.e. $0 < P_x < \infty$. Note that a signal cannot be both an energy and a power signal simultaneously. The energy signals have zero average power whereas the power signals have infinite total energy. Some signals, however, can be classified as neither power signals nor as energy signals. For example, the signal $e^{2t}u(t)$ is a growing exponential whose average power cannot be calculated. Such signals are generally of little interest

Example 1.6

Consider the CT signals shown in Figs. 1.9(a) and (b). Calculate the instantaneous power, average power, and energy present in the two signals. Classify these signals as power or energy signals.



(a)

Solution

(a) The signal $x(t)$ can be expressed as follows:

$$x(t) = \begin{cases} 5 & -2 \le t \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

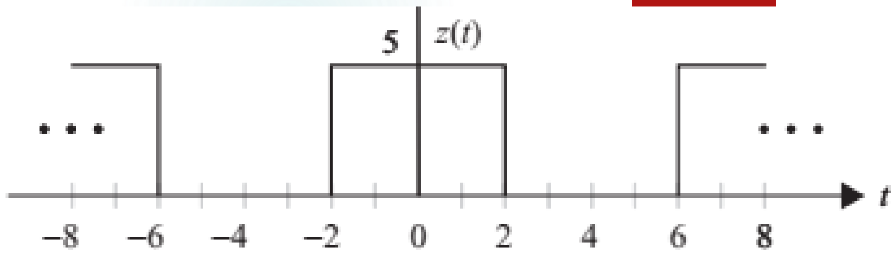
instantaneous power $P_x(t) = \begin{cases} 25 & -2 \le t \le 2 \\ 0 & \text{otherwise;} \end{cases}$

energy $E_x = \int\limits_{-\infty}^{\infty} |x(t)|^2 \mathrm{d}t = \int\limits_{-2}^{2} 25\, \mathrm{d}t = 100;$

average power $P_x = \lim\limits_{T \to \infty} \frac{1}{T} E_x = 0.$

Because $x(t)$ has finite energy ($0 < E_x = 100 < \infty$) it is an energy signal.

(b) The signal $z(t)$ is a periodic signal with fundamental period 8 and over one period is expressed as follows:



(b)

$$z(t) = \begin{cases} 5 & -2 \le t \le 2 \\ 0 & 2 < |t| \le 4, \end{cases}$$

with $z(t+8) = z(t)$. The instantaneous power, average power, and energy of the signal are calculated as follows:

instantaneous power $$P_z(t) = \begin{cases} 25 & -2 \le t \le 2 \\ 0 & 2 < |t| \le 4 \end{cases} \quad \text{and } P_z(t+8) = P_z(t);$$

average power $$P_z = \frac{1}{8}\int_{-4}^{4} |z(t)|^2\, dt = \frac{1}{8}\int_{-2}^{2} 25\, dt = \frac{100}{8} = 12.5;$$

energy $$E_z = \int_{-\infty}^{\infty} |z(t)|^2\, dt = \infty.$$

Because the signal has finite power ($0 < P_z = 12.5 < \infty$), $z(t)$ is a power signal.

Sinusoidal Signals

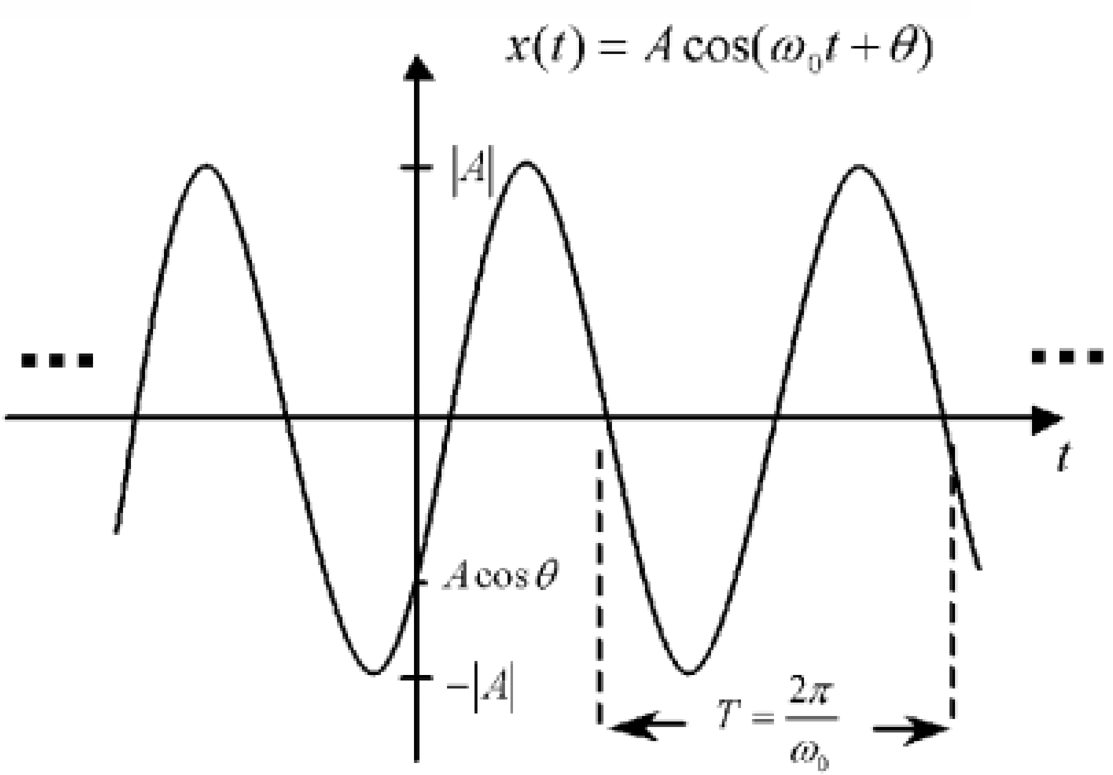
$$x(t) = A\cos(\omega t + \phi),$$

where A is the amplitude, ω is the frequency in radians per second, and ϕ is the phase angle in radians. Figure 1.31(a) presents the waveform of a sinusoidal signal for $A = 4$ and $\phi = +\pi/6$. A sinusoidal signal is an example of a periodic signal, the period of which is

$$T = \frac{2\pi}{\omega}.$$

We may readily show that this is the period for the sinusoidal signal of Eq. (1.35) by writing

$$\begin{aligned} x(t+T) &= A\cos(\omega(t+T)+\phi) \\ &= A\cos(\omega t + \omega T + \phi) \\ &= A\cos(\omega t + 2\pi + \phi) \\ &= A\cos(\omega t + \phi) \\ &= x(t), \end{aligned}$$



Consider next the discrete-time version of a sinusoidal signal, written as

$$x[n] = A\cos(\Omega n + \phi).$$

This discrete-time signal may or may not be periodic. For it to be periodic with a period of, say, N samples, it must satisfy Eq. (1.10) for all integer n and some integer N. Substituting $n + N$ for n in Eq. (1.39) yields

$$x[n + N] = A\cos(\Omega n + \Omega N + \phi).$$

For Eq. (1.10) to be satisfied, in general, we require that

$$\Omega N = 2\pi m \qquad \text{radians,}$$

or

$$\Omega = \frac{2\pi m}{N} \text{ radians/cycle,} \qquad \text{integer } m, N. \tag{1.40}$$

The important point to note here is that, unlike continuous-time sinusoidal signals, not all discrete-time sinusoidal systems with arbitrary values of Ω are periodic.

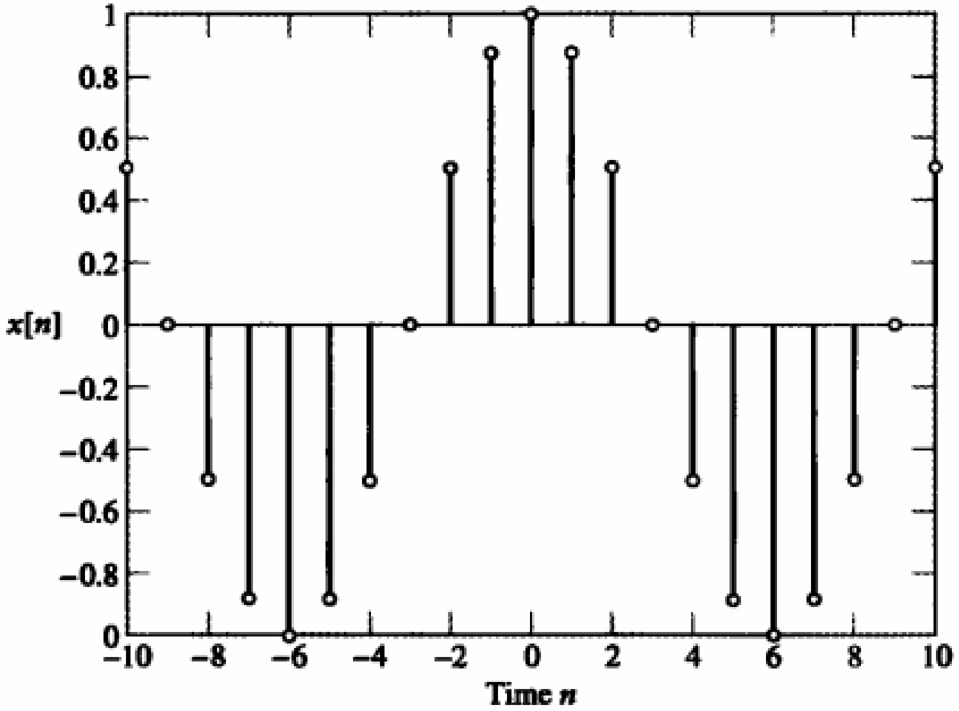


FIGURE 1.33 Discrete-time sinusoidal signal.

EXAMPLE 1.7 DISCRETE-TIME SINUSOIDAL SIGNALS A pair of sinusoidal signals with a common angular frequency is defined by

$$x_1[n] = \sin[5\pi n]$$

and

$$x_2[n] = \sqrt{3}\cos[5\pi n].$$

(a) Both $x_1[n]$ and $x_2[n]$ are periodic. Find their common fundamental period.

(b) Express the composite sinusoidal signal

$$y[n] = x_1[n] + x_2[n]$$

in the form $y[n] = A\cos(\Omega n + \phi)$, and evaluate the amplitude A and phase ϕ.

Solution:

(a) The angular frequency of both $x_1[n]$ and $x_2[n]$ is

$$\Omega = 5\pi \text{ radians/cycle.}$$

Solving Eq. (1.40) for the period N, we get

$$N = \frac{2\pi m}{\Omega} = \frac{2\pi m}{5\pi} = \frac{2m}{5}.$$

For $x_1[n]$ and $x_2[n]$ to be periodic, N must be an integer. This can be so only for $m = 5, 10, 15, \ldots$, which results in $N = 2, 4, 6, \ldots$.

(b) Express the composite sinusoidal signal

$$y[n] = x_1[n] + x_2[n]$$

in the form $y[n] = A\cos(\Omega n + \phi)$, and evaluate the amplitude A and phase ϕ.

(b) Recall the trigonometric identity

$$A\cos(\Omega n + \phi) = A\cos(\Omega n)\cos(\phi) - A\sin(\Omega n)\sin(\phi).$$

Letting $\Omega = 5\pi$, we see that the right-hand side of this identity is of the same form as $x_1[n] + x_2[n]$. We may therefore write

$$A\sin(\phi) = -1 \quad \text{and} \quad A\cos(\phi) = \sqrt{3}.$$

Hence,

$$\tan(\phi) = \frac{\sin(\phi)}{\cos(\phi)} = \frac{\text{amplitude of } x_1[n]}{\text{amplitude of } x_2[n]}$$
$$= \frac{-1}{\sqrt{3}},$$

rom which we find that $\phi = -\pi/3$ radians. Substituting this value into the equation

$$A\sin(\phi) = -1$$

nd solving for the amplitude A, we get

$$A = -1/\sin\left(-\frac{\pi}{3}\right)$$
$$= 2.$$

ccordingly, we may express $y[n]$ as

$$y[n] = 2\cos\left(5\pi n - \frac{\pi}{3}\right).$$ ■

Real Exponential Signals

Real exponential signals can be defined both in continuous time and in discrete time.

Continuous Time

We can define a general real exponential signal as follows:

$$x(t) = Ce^{\alpha t}, \quad 0 \neq C, \alpha \in \mathbb{R}.$$

We now look at different cases depending on the value of parameter α.

$\boxed{\text{Case } \alpha = 0:}$ We simply get the constant signal $x(t) = C$.

$\boxed{\text{Case } \alpha > 0:}$ The exponential tends to infinity as $t \to +\infty$, as shown in Figure 1.16, where $C > 0$. Notice that $x(0) = C$.

$\boxed{\text{Case } \alpha < 0:}$ The exponential tends to zero as $t \to +\infty$; see Figure 1.17, where $C < 0$.

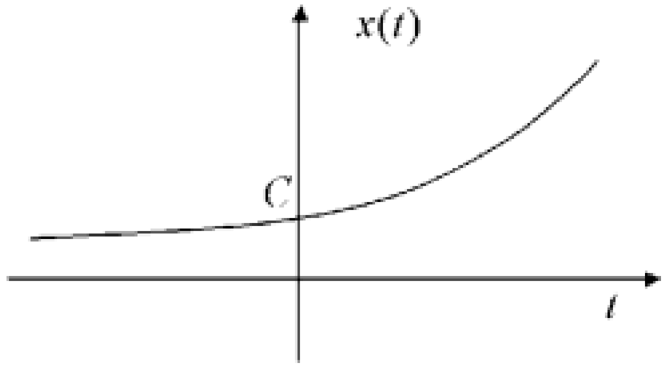


FIGURE 1.16 Continuous-time exponential signal growing unbounded with time.

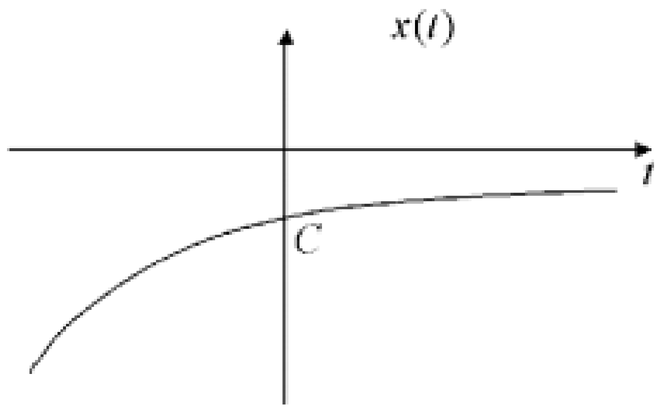


FIGURE 1.17 Continuous-time exponential signal tapering off to zero with time.

For a physical example of an exponential signal, consider a so-called lossy capacitor, as depicted in Fig. 1.29. The capacitor has capacitance C, and the loss is represented by shunt resistance R. The capacitor is charged by connecting a battery across it, and then the battery is removed at time $t = 0$. Let V_0 denote the initial value of the voltage developed across the capacitor. From the figure, we readily see that the operation of the capacitor for $t \geq 0$ is described by

$$RC\frac{d}{dt}v(t) + v(t) = 0, \tag{1.32}$$

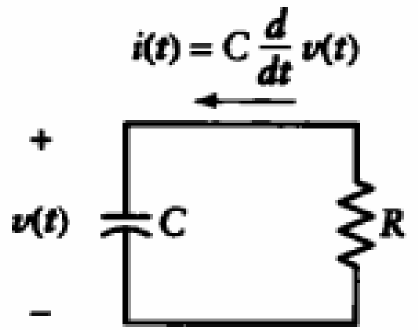


FIGURE 1.29 Lossy capacitor, with the loss represented by shunt resistance R.

where $v(t)$ is the voltage measured across the capacitor at time t. Equation (1.32) is a *differential equation of order one*. Its solution is given by

$$v(t) = V_0 e^{-t/(RC)}, \tag{1.33}$$

Discrete Time

We define a general real discrete-time exponential signal as follows:

$$x[n] = C\alpha^n, \quad C, \alpha \in \mathbb{R}.$$

There are six cases to consider, apart from the trivial cases $\alpha = 0$ or $C = 0$: $\alpha = 1$, $\alpha > 1$, $0 < \alpha < 1$, $\alpha < -1$, $\alpha = -1$, and $-1 < \alpha < 0$. Here we assume that $C > 0$, but for C negative, the graphs would simply be flipped images of the ones given around the time axis.

Case $\alpha = 1$: We get a constant signal $x[n] = C$.

Case $\alpha > 1$: We get a positive signal that grows exponentially, as shown in Figure 1.18.

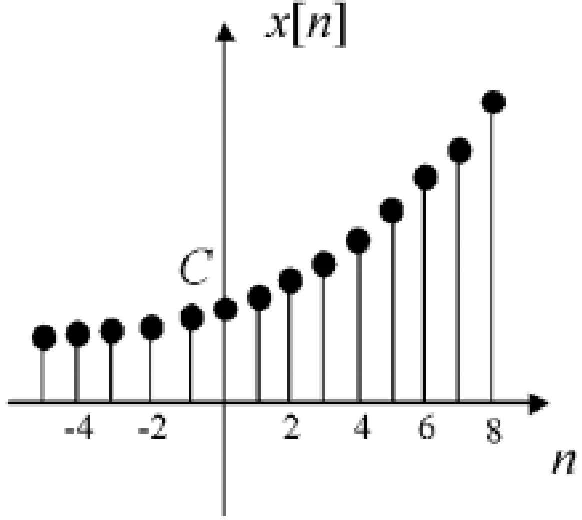


FIGURE 1.18 Discrete-time exponential signal growing unbounded with time.

Case $0<\alpha<1$: The signal $x[n]=C\alpha^n$ is positive and decays exponentially, as shown in Figure 1.19.

Case $\alpha<-1$: The signal $x[n]=C\alpha^n$ alternates between positive and negative values and grows exponentially in magnitude with time. This is shown in Figure 1.20.

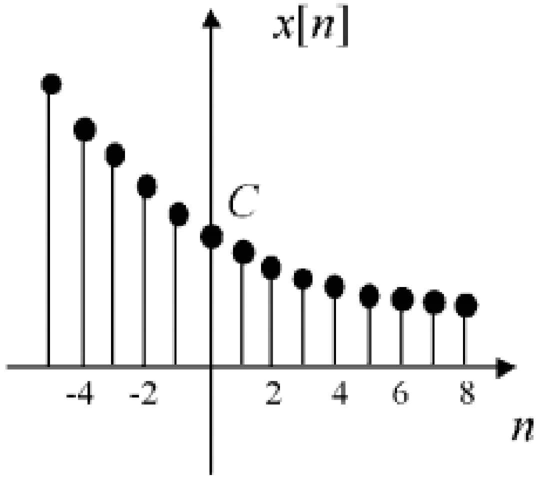


FIGURE 1.19 Discrete-time exponential signal tapering off to zero with time.

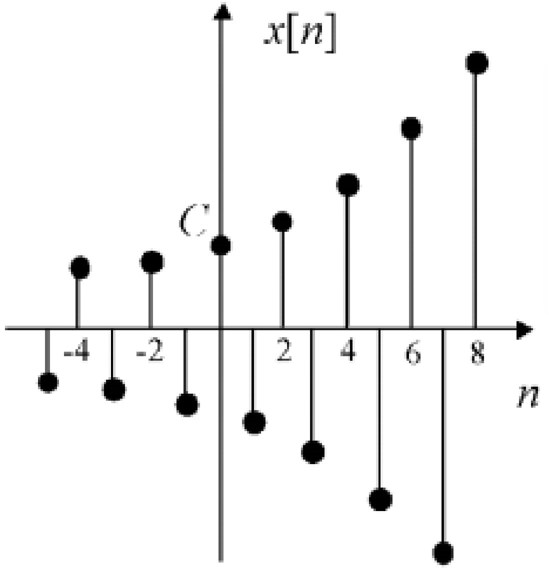


FIGURE 1.20 Discrete-time exponential signal alternating and growing unbounded with time.

Case $\alpha = -1$: The signal alternates between C and $-C$, as seen in Figure 1.21.

Case $-1 < \alpha < 0$: The signal alternates between positive and negative values and decays exponentially in magnitude with time, as shown in Figure 1.22.

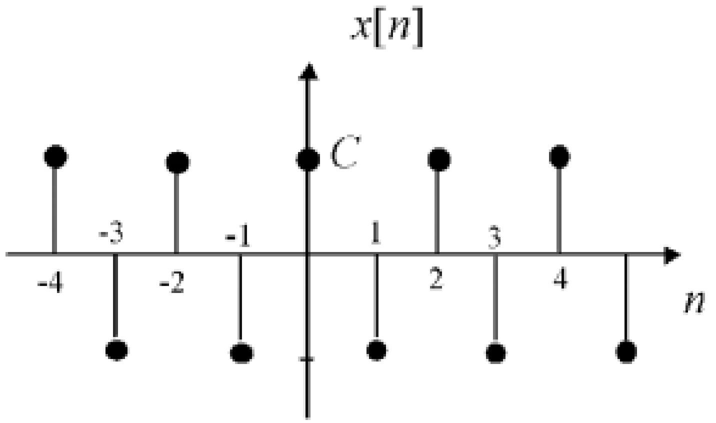


FIGURE 1.21 Discrete-time exponential signal reduced to an alternating periodic signal.

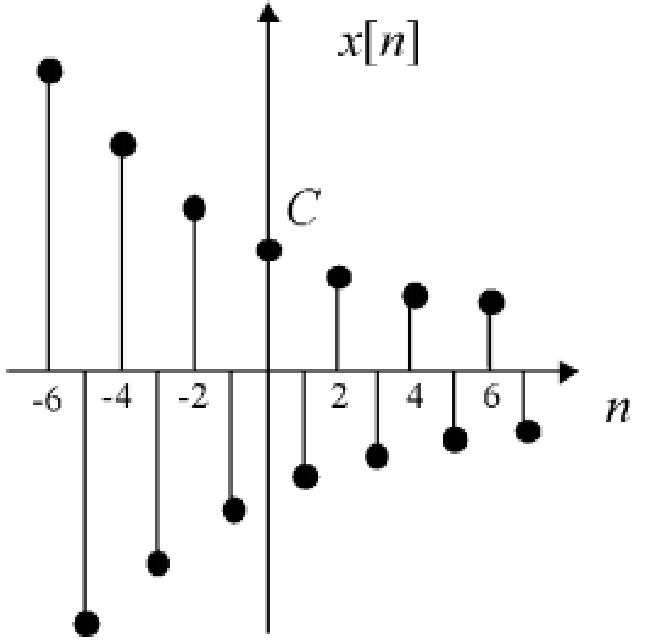


FIGURE 1.22 Discrete-time exponential signal alternating and tapering off to zero with time.

Complex Exponential Signals

Complex exponential signals can also be defined both in continuous time and in discrete time. They have real and imaginary parts with sinusoidal behavior.

Continuous Time

The continuous-time complex exponential signal can be defined as follows:

$$x(t) := Ce^{at}, \quad C, a \in \mathbb{C}, \tag{1.12}$$

where $C = Ae^{j\theta}$, $A, \theta \in \mathbb{R}, A > 0$ is expressed in polar form, and $a = \alpha + j\omega_0$, $\alpha, \omega_0 \in \mathbb{R}$ is expressed in rectangular form. Thus, we can write

$$\begin{aligned} x(t) &= Ae^{j\theta} e^{(\alpha + j\omega_0)t} \\ &= Ae^{\alpha t} e^{j(\omega_0 t + \theta)} \end{aligned} \tag{1.13}$$

If we look at the second part of Equation 1.13, we can see that $x(t)$ represents either a circular or a spiral trajectory in the complex plane, depending whether α is zero, negative, or positive. The term $e^{j(\omega_0 t+\theta)}$ describes a unit circle centered at the origin counterclockwise in the complex plane as time varies from $t=-\infty$ to $t=+\infty$, as shown in Figure 1.23 for the case $\theta = 0$. The times t_k indicated in the figure are the times when the complex point $e^{j\omega_0 t_k}$ has a phase of $\pi/4$.

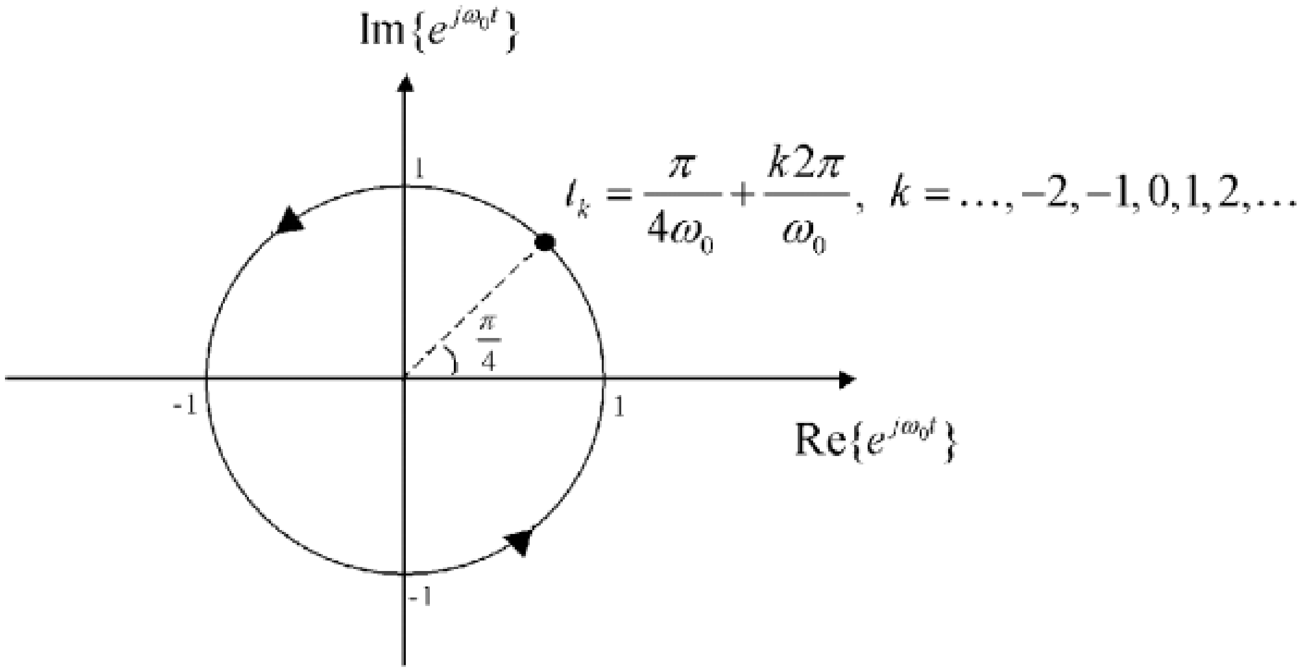


FIGURE 1.23 Trajectory described by the complex exponential.

Using Euler's relation, we obtain the signal in rectangular form:

$$x(t) = Ae^{\alpha t}\cos(\omega_0 t + \theta) + jAe^{\alpha t}\sin(\omega_0 t + \theta), \tag{1.14}$$

where $\mathrm{Re}\{x(t)\} = Ae^{\alpha t}\cos(\omega_0 t + \theta)$ and $\mathrm{Im}\{x(t)\} = Ae^{\alpha t}\sin(\omega_0 t + \theta)$ are the real part and imaginary part of the signal, respectively. Both are sinusoidal, with time-varying amplitude (or envelope) $Ae^{\alpha t}$. We can see that the exponent $\alpha = \mathrm{Re}\{a\}$ defines the type of real and imaginary parts we get for the signal.

For the case $\alpha = 0$, we obtain a complex periodic signal of period $T = \frac{2\pi}{\omega_0}$ (as shown in Figure 1.23 but with radius A) whose real and imaginary parts are sinusoidal:

$$x(t) = A\cos(\omega_0 t + \theta) + jA\sin(\omega_0 t + \theta). \tag{1.15}$$

The real part of this signal is shown in Figure 1.24.

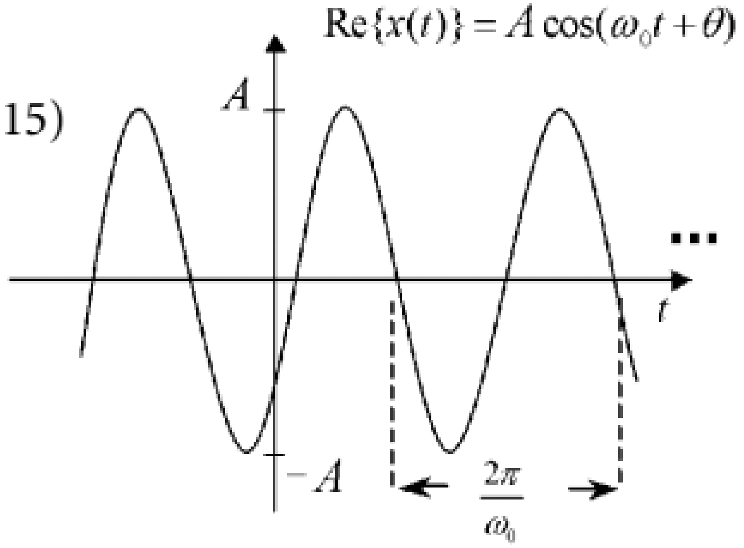


FIGURE 1.24 Real part of periodic complex exponential for $\alpha = 0$.

For the case $\alpha < 0$, we get a complex periodic signal multiplied by a decaying exponential. The real and imaginary parts are *damped sinusoids* that are signals that can describe, for example, the response of an *RLC* (resistance-inductance-capacitance) circuit or the response of a mass-spring-damper system such as a car suspension. The real part of $x(t)$ is shown in Figure 1.25.

For the case $\alpha > 0$, we get a complex periodic signal multiplied by a growing exponential. The real and imaginary parts are *growing sinusoids* that are signals that can describe the response of an unstable feedback control system. The real part of $x(t)$ is shown in Figure 1.26.

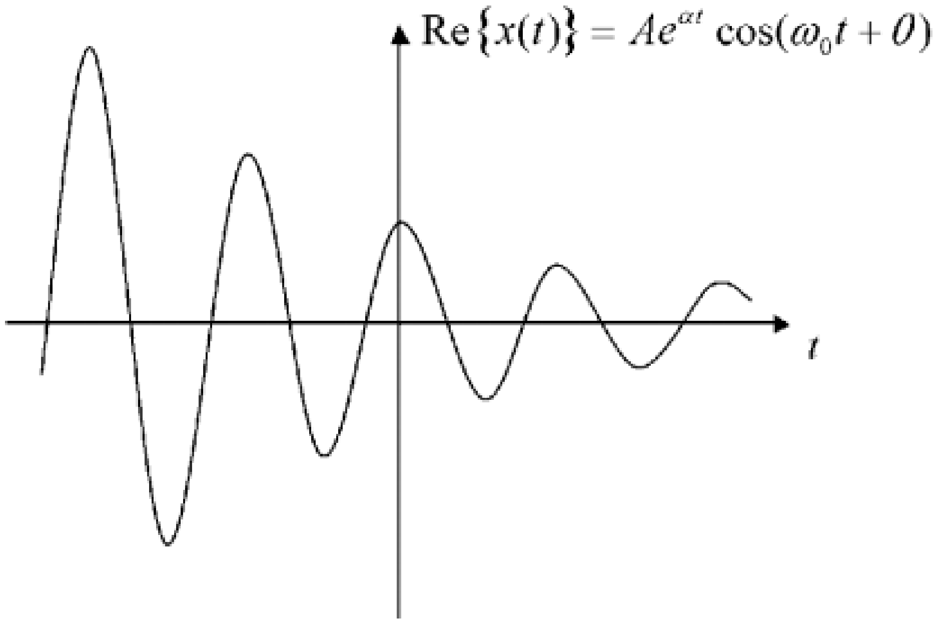


FIGURE 1.25 Real part of damped complex exponential for $\alpha < 0$.

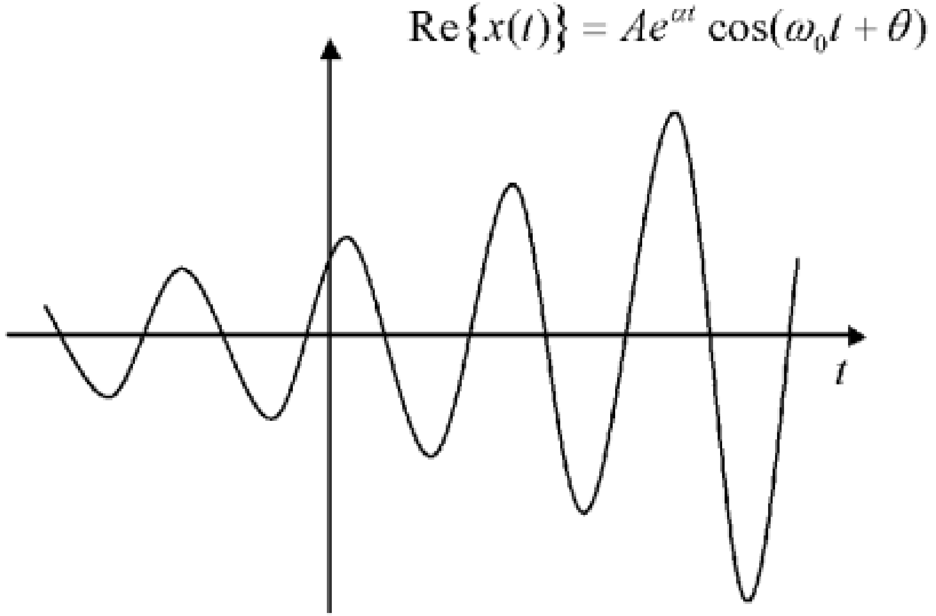


FIGURE 1.26 Real part of growing complex exponential for $\alpha > 0$.

Discrete Time

The discrete-time complex exponential signal can be defined as follows:

$$x[n] = Ca^n, \tag{1.16}$$

where $C, a \in \mathbb{C}, \quad C = Ae^{j\theta}, A, \theta \in \mathbb{R}, A > 0 \quad a = re^{j\omega_0}, r, \omega_0 \in \mathbb{R}, r > 0$.

Substituting the polar forms of C and a in Equation 1.16, we obtain a useful expression for $x[n]$ with time-varying amplitude:

$$\begin{aligned} x[n] &= Ae^{j\theta} r^n e^{j\omega_0 n} \\ &= Ar^n e^{j(\omega_0 n + \theta)}, \end{aligned} \tag{1.17}$$

and using Euler's relation, we get the rectangular form of the discrete-time complex exponential:

$$x[n] = Ar^n \cos(\omega_0 n + \theta) + jAr^n \sin(\omega_0 n + \theta). \tag{1.18}$$

Clearly, the magnitude r of a determines whether the envelope of $x[n]$ grows, decreases, or remains constant with time.

For the case $r = 1$, we obtain a complex signal whose real and imaginary parts have a sinusoidal envelope (they are sampled cosine and sine waves), *but the signal is not necessarily periodic*! We will discuss this issue in the next section.

$$x[n] = A\cos(\omega_0 n + \theta) + jA\sin(\omega_0 n + \theta) \tag{1.19}$$

Figure 1.27 shows the real part of a complex exponential signal with $r = 1$.

For the case $r < 1$, we get a complex signal whose real and imaginary parts are damped sinusoidal signals (see Figure 1.28).

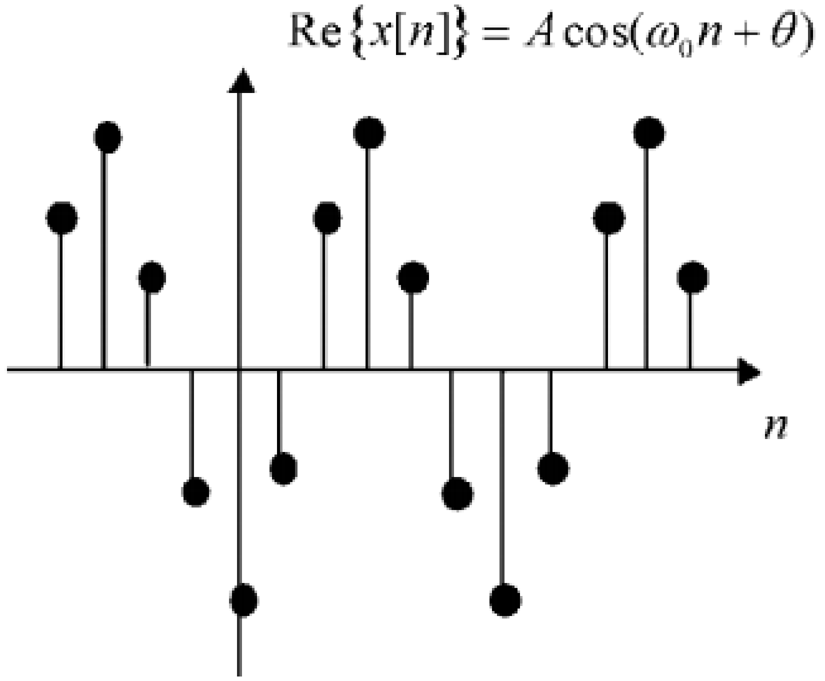


FIGURE 1.27 Real part of discrete-time complex exponential for $r = 1$.

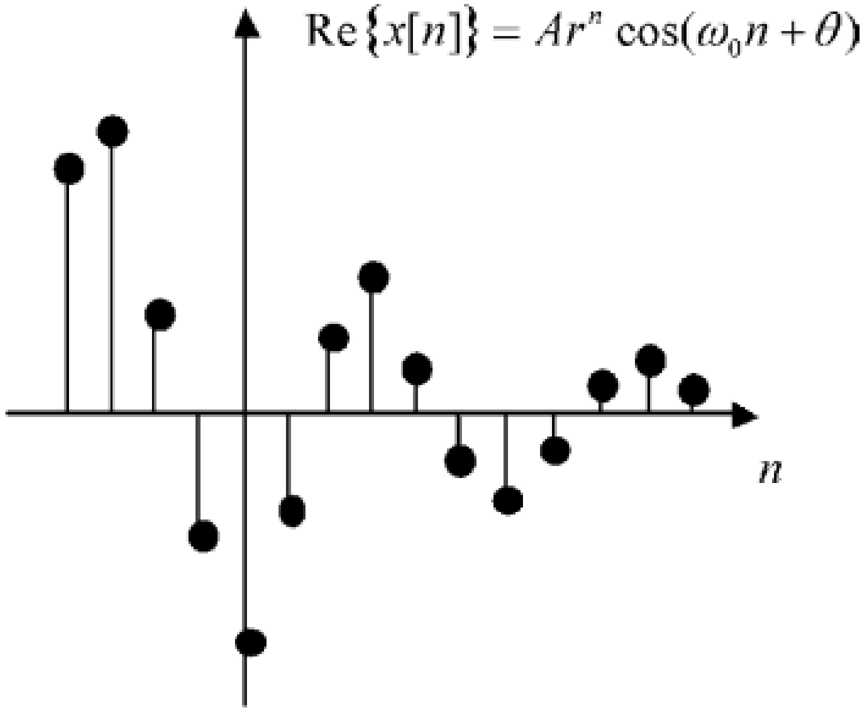


FIGURE 1.28 Real part of discrete-time damped complex exponential for $r < 1$.

For the case r > 1, we obtain a complex signal whose real and imaginary parts are growing sinusoidal sequences, as shown in Figure 1.29.

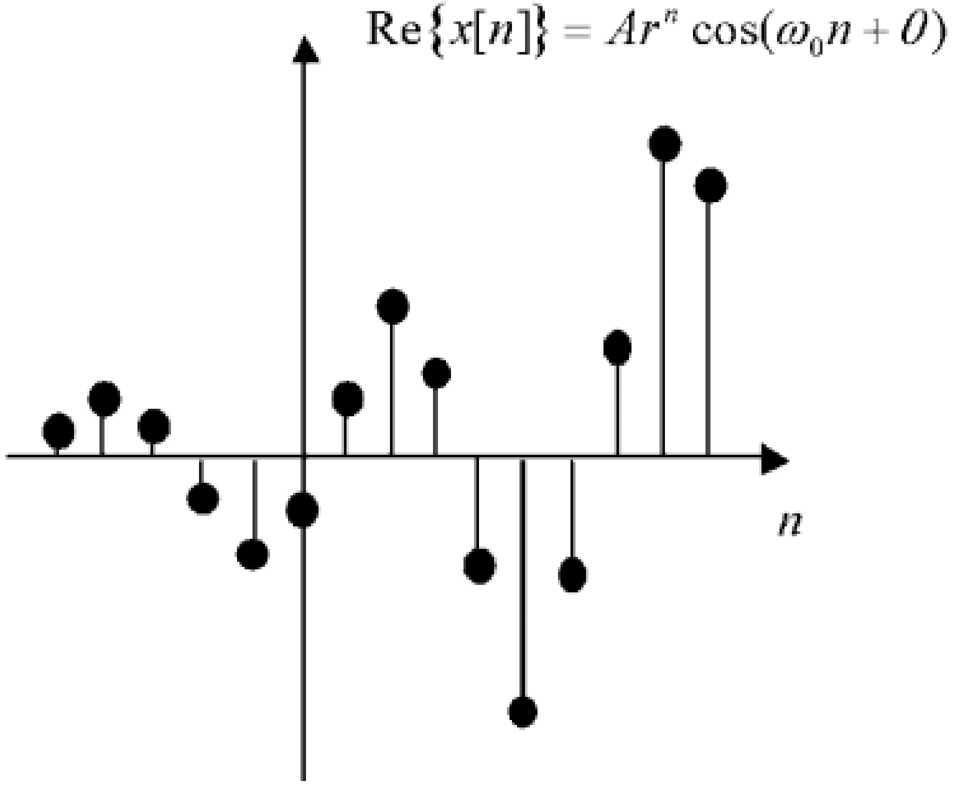


FIGURE 1.29 Real part of growing complex exponential for $r > 1$.

Causal exponential function

In practical signal processing applications, input signals start at time $t = 0$. Signals that start at $t = 0$ are referred to as causal signals. The causal exponential function is given by

$$x(t) = e^{st}u(t) = \begin{cases} e^{st} & t \geq 0 \\ 0 & t < 0, \end{cases} \tag{1.41}$$

where we have used the unit step function to incorporate causality in the complex exponential functions. Similarly, the causal implementation of the DT exponential function is defined as follows:

$$x[k] = e^{sk}u[k] = \begin{cases} e^{sk} & k \geq 0 \\ 0 & k < 0. \end{cases} \tag{1.42}$$

The same concept can be extended to derive causal implementations of sinusoidal and other non-causal signals.

Sinc function

The CT sinc function is defined as follows:

$$\mathrm{sinc}(\omega_0 t) = \frac{\sin(\pi \omega_0 t)}{\pi \omega_0 t}, \qquad (1.36)$$

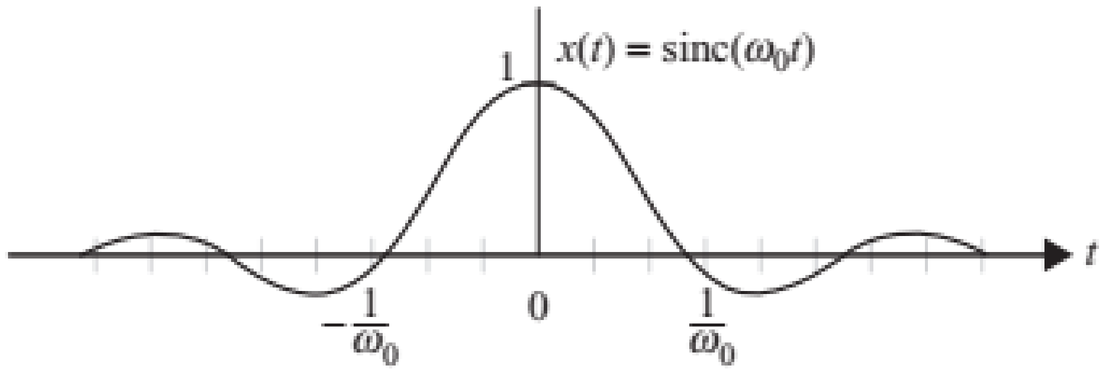
which is plotted in Fig. 1.12(k). In some text books, the sinc function is alternatively defined as follows:

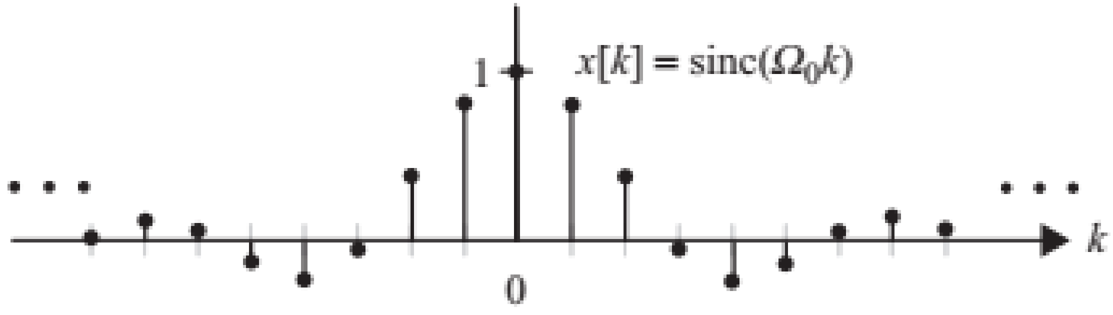
$$\mathrm{sinc}(\omega_0 t) = \frac{\sin(\omega_0 t)}{\omega_0 t}.$$

$$\mathrm{sinc}\,(x) \equiv \begin{cases} 1 & \text{for } x = 0 \\ \dfrac{\sin x}{x} & \text{otherwise,} \end{cases}$$

In this text, we will use the definition in Eq. (1.36) for the sinc function. The DT sinc function is defined as follows:

$$\mathrm{sinc}(\Omega_0 k) = \frac{\sin(\pi \Omega_0 k)}{\pi \Omega_0 k}, \qquad (1.37)$$





STEP FUNCTION

The discrete-time version of the unit-step function is defined by

$$\boxed{u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases},}$$

which is illustrated in Fig. 1.37.

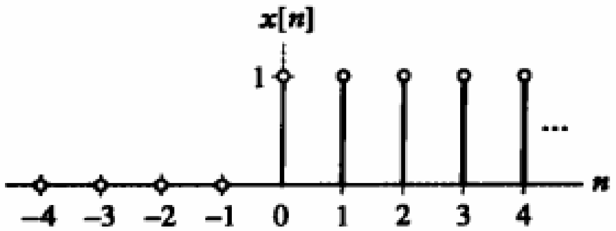


FIGURE 1.37 Discrete-time version of step function of unit amplitude.

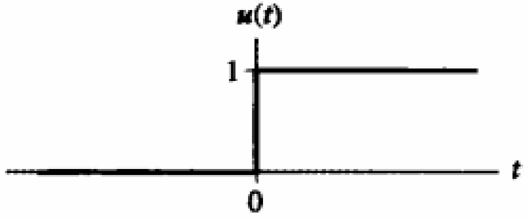


FIGURE 1.38 Continuous-time version of the unit-step function of unit amplitude.

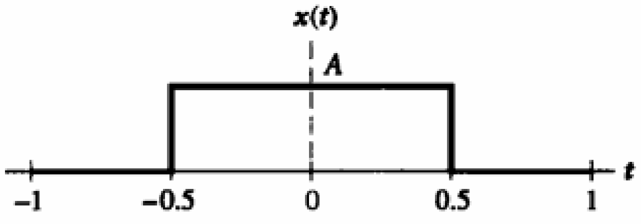
The continuous-time version of the unit-step function is defined by

$$\boxed{u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}.}$$

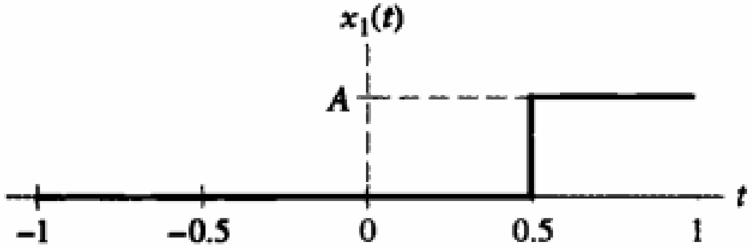
Note that since $u(t)$ is discontinuous at the origin, it cannot be formally differentiated. We will nonetheless define the derivative of the step signal later and give its interpretation.

One of the uses of the step signal is to apply it at the input of a system in order to characterize its behavior. The resulting output signal is called the *step response* of the system. Another use is to truncate some parts of a signal by multiplication with time-shifted unit step signals.

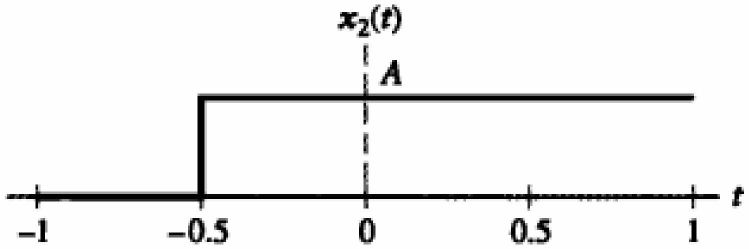
EXAMPLE 1.8 RECTANGULAR PULSE Consider the rectangular pulse $x(t)$ shown in Fig. 1.39(a). This pulse has an amplitude A and duration of 1 second. Express $x(t)$ as a weighted sum of two step functions.



(a)



(b)



(c)

$$x(t) = x_2(t) - x_1(t).$$

Solution: The rectangular pulse $x(t)$ may be written in mathematical terms as

$$x(t) = \begin{cases} A, & 0 \le |t| < 0.5 \\ 0, & |t| > 0.5 \end{cases},$$

where $|t|$ denotes the magnitude of time t. The rectangular pulse $x(t)$ is represented as the difference of two time-shifted step functions, $x_1(t)$ and $x_2(t)$, which are defined in Figs. 1.39(b) and 1.39(c), respectively. On the basis of this figure, we may express $x(t)$ as

$$x(t) = Au\left(t + \frac{1}{2}\right) - Au\left(t - \frac{1}{2}\right), \tag{1.56}$$

where $u(t)$ is the unit-step function. ■

▶ **Problem 1.22** A discrete-time signal

$$x[n] = \begin{cases} 1, & 0 \le n \le 9 \\ 0, & \text{otherwise} \end{cases}.$$

Using $u[n]$, describe $x[n]$ as the superposition of two step functions.

Answer: $x[n] = u[n] - u[n - 10]$.

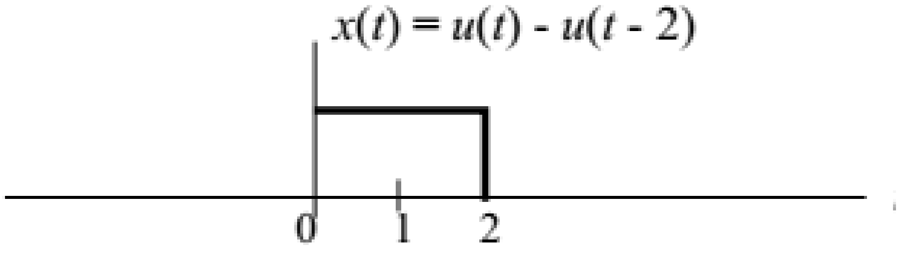
1.54 Sketch the waveforms of the following signals:

(a) $x(t) = u(t) - u(t-2)$

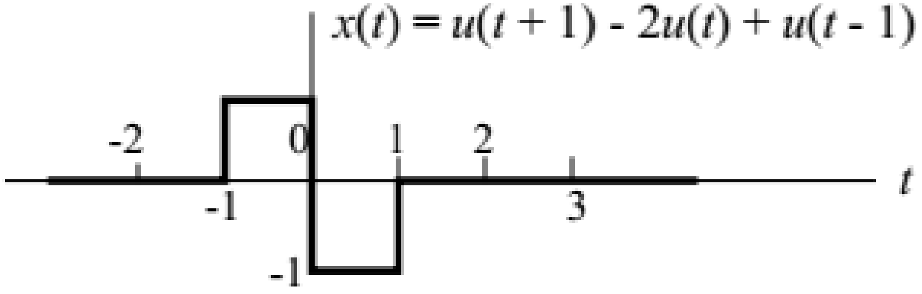
(b) $x(t) = u(t+1) - 2u(t) + u(t-1)$

(c) $x(t) = -u(t+3) + 2u(t+1) - 2u(t-1) + u(t-3)$

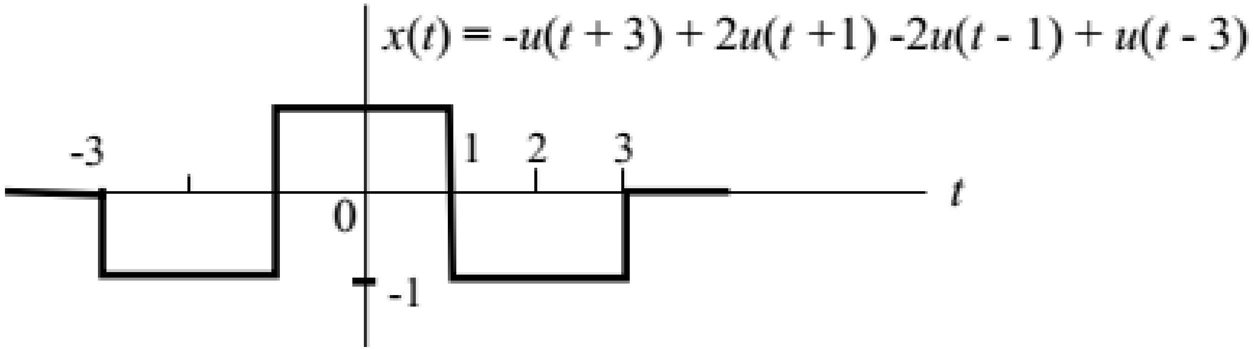
(a)



(b)



(c)



Impulse Function

The continuous-time version of the unit impulse is defined by the following pair of relations:

$$\boxed{\delta(t) = 0 \quad \text{for} \quad t \neq 0} \tag{1.59}$$

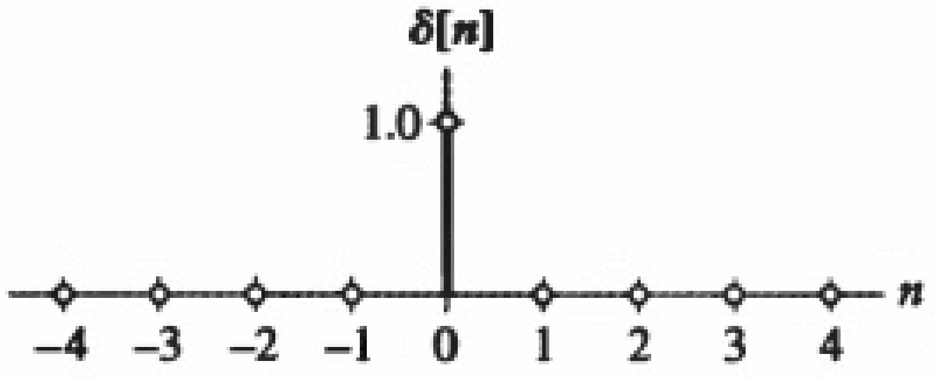
and

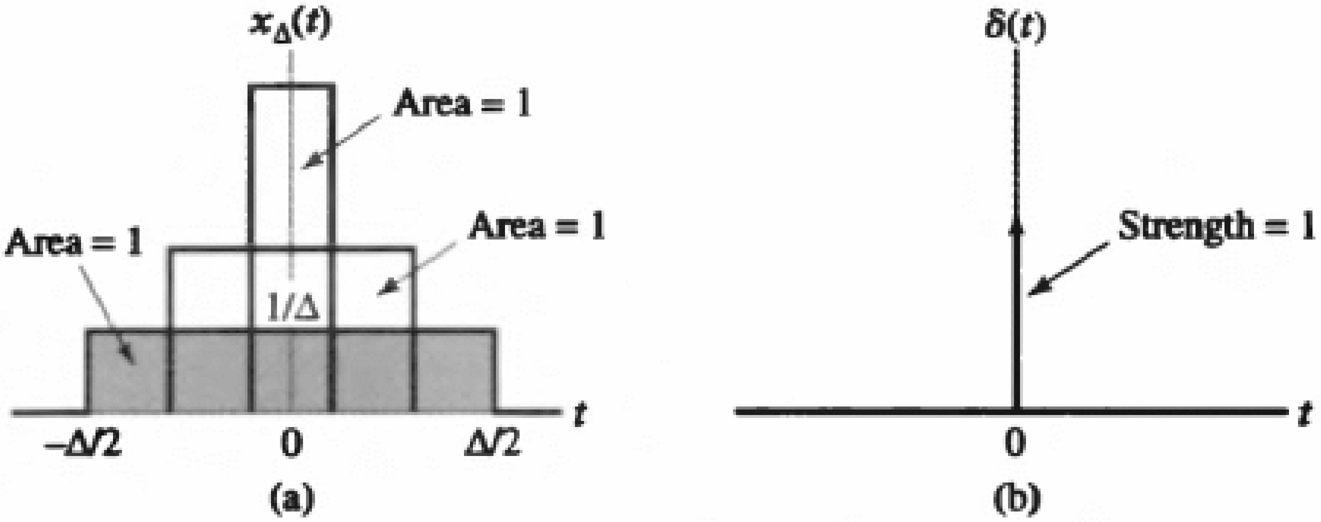
$$\boxed{\int_{-\infty}^{\infty} \delta(t)\, dt = 1.} \tag{1.60}$$

Equation (1.59) says that the impulse $\delta(t)$ is zero everywhere except at the origin. Equation (1.60) says that the total area under the unit impulse is unity. The impulse $\delta(t)$ is also referred to as the *Dirac delta function*.

The discrete-time version of the *unit impulse* is defined by

$$\boxed{\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}.}$$





The impulse $\delta(t)$ and the unit-step function $u(t)$ are related to each other in that if we are given either one, we can uniquely determine the other. Specifically, $\delta(t)$ is the derivative of $u(t)$ with respect to time t, or

$$\delta(t) = \frac{d}{dt}u(t). \tag{1.62}$$

Conversely, the step function $u(t)$ is the integral of the impulse $\delta(t)$ with respect to time t:

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau \tag{1.63}$$

Properties of impulse function

(i) The impulse function is an even function, i.e. $\delta(t) = \delta(-t)$.

(ii) Integrating a unit impulse function results in one, provided that the limits of integration enclose the origin of the impulse. Mathematically,

$$\int_{-T}^{T} A\delta(t - t_0)\mathrm{d}t = \begin{cases} A & \text{for } -T < t_0 < T \\ 0 & \text{elsewhere.} \end{cases} \tag{1.44}$$

(iii) The scaled and time-shifted version $\delta(at + b)$ of the unit impulse function is given by

$$\delta(at + b) = \frac{1}{a}\delta\left(t + \frac{b}{a}\right). \tag{1.45}$$

(iv) When an arbitrary function $\phi(t)$ is multiplied by a shifted impulse function, the product is given by

$$\phi(t)\delta(t - t_0) = \phi(t_0)\delta(t - t_0). \tag{1.46}$$

In other words, multiplication of a CT function and an impulse function produces an impulse function, which has an area equal to the value of the CT function at the location of the impulse. Combining properties (ii) and (iv), it is straightforward to show that

$$\int_{-\infty}^{\infty} \phi(t)\delta(t - t_0)\mathrm{d}t = \phi(t_0). \tag{1.47}$$

Example 1.12

Simplify the following expressions:

(i) $\frac{5-jt}{7+t^2}\delta(t)$;

(ii) $\int_{-\infty}^{\infty}(t+5)\delta(t-2)dt$;

(iii) $\int_{-\infty}^{\infty}e^{j0.5\pi\omega+2}\delta(\omega-5)d\omega$.

Solution

(i) Using Eq. (1.46) yields $\frac{5-jt}{7+t^2}\delta(t)=\left[\frac{5-jt}{7+t^2}\right]_{t=0}\delta(t)=\frac{5}{7}\delta(t)$.

(ii) Using Eq. (1.46) yields

$$\int_{-\infty}^{\infty}(t+5)\delta(t-2)dt=\int_{-\infty}^{\infty}[(t+5)]_{t=2}\delta(t-2)dt=7\int_{-\infty}^{\infty}\delta(t-2)dt.$$

Since the integral computes the area enclosed by the unit step function, which is one, we obtain

$$\int_{-\infty}^{\infty}(t+5)\delta(t-2)dt=7\int_{-\infty}^{\infty}\delta(t-2)dt=7.$$

(iii) Using Eq. (1.46) yields

$$\int_{-\infty}^{\infty}e^{j0.5\pi\omega+2}\delta(\omega-5)d\omega=\int_{-\infty}^{\infty}[e^{j0.5\pi\omega+2}]_{\omega=5}\delta(\omega-5)d\omega$$

$$=e^{j2.5\pi+2}\int_{-\infty}^{\infty}\delta(\omega-5)d\omega.$$

Since $\exp(j2.5\pi+2)=j\exp(2)$ and the integral equals one, we obtain

$$\int_{-\infty}^{\infty}e^{j0.5\pi\omega+2}\delta(\omega-5)d\omega=je^2.$$

Ramp Function

The impulse function $\delta(t)$ is the derivative of the step function $u(t)$ with respect to time. By the same token, the integral of the step function $u(t)$ is a ramp function of unit slope. This latter test signal is formally defined as

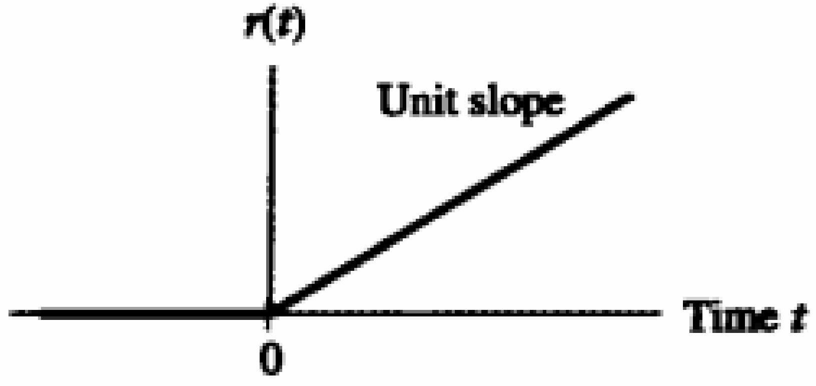
$$r(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}. \tag{1.74}$$

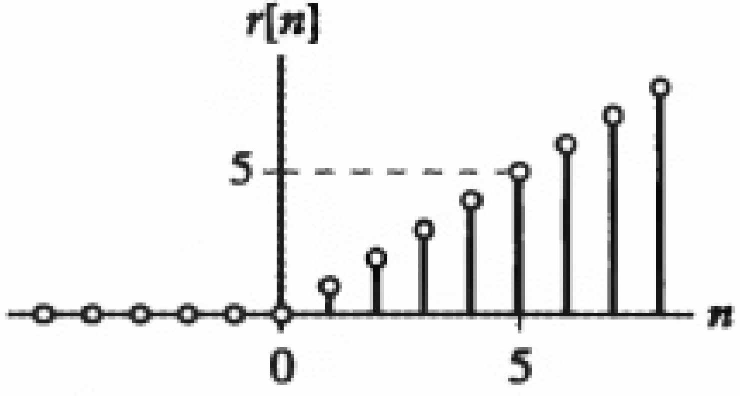
Equivalently, we may write

$$r(t) = tu(t). \tag{1.75}$$

The discrete-time version of the ramp function is defined by

$$r[n] = \begin{cases} n, & n \geq 0 \\ 0, & n < 0 \end{cases},$$





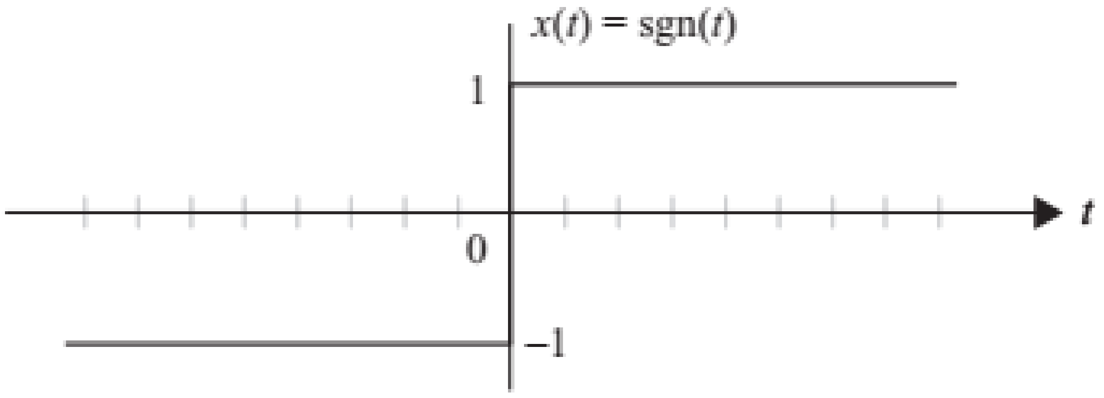
Signum function

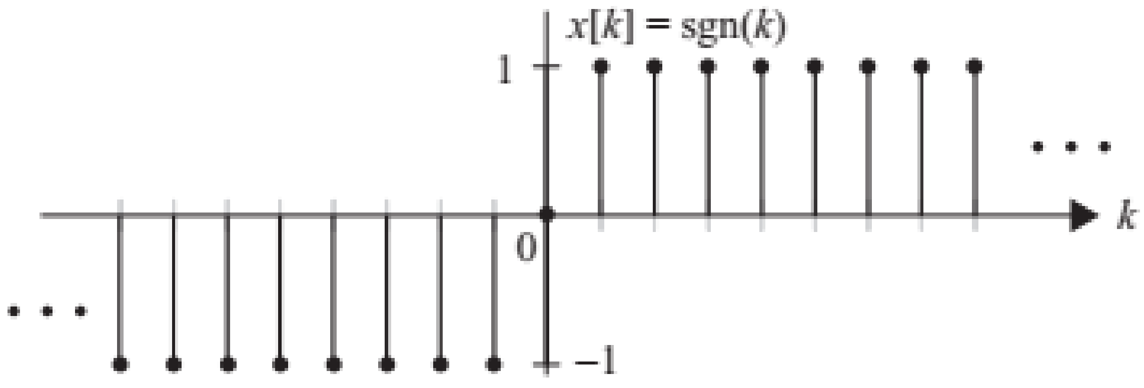
The *signum* (or *sign*) function, denoted by sgn(t), is defined as follows:

$$\operatorname{sgn}(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0. \end{cases} \qquad (1.30)$$

function, denoted by sgn(k), is defined as follows:

$$\operatorname{sgn}[k] = \begin{cases} 1 & k > 0 \\ 0 & k = 0 \\ -1 & k < 0 \end{cases}$$





SÜREKLİ ZAMANLI VE AYRIK ZAMANLI SİSTEMLER

In mathematical terms, a system may be viewed as an *interconnection of operations* that transforms an input signal into an output signal with properties different from those of the input signal. The signals may be of the continuous-time or discrete-time variety or a mixture of both. Let the overall *operator H* denote the action of a system. Then the application of a continuous-time signal $x(t)$ to the input of the system yields the output signal

$$y(t) = H\{x(t)\}. \tag{1.78}$$

Figure 1.49(a) shows a block diagram representation of Eq. (1.78). Correspondingly, for the discrete-time case, we may write

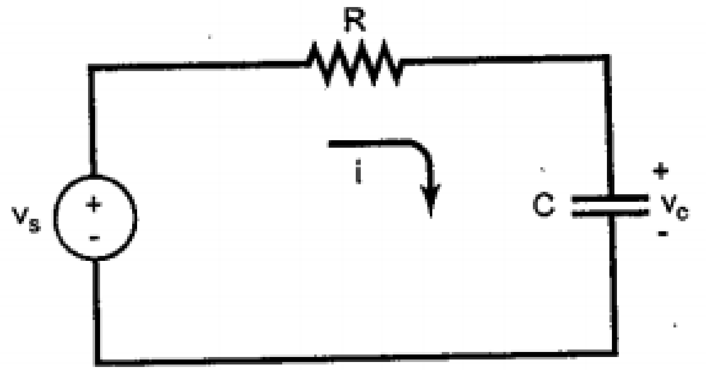
$$y[n] = H\{x[n]\}, \tag{1.79}$$

FIGURE 1.49 Block diagram representation of operator H for (a) continuous time and (b) discrete time.

Örnek 1.8

Şekil 1.1'de gösterilen RC devresini ele alalım. Eğer $v_s(t)$'yi giriş sinyali ve $v_c(t)$'yi çıkış sinyali olarak alırsak, böylece basit devre analizini giriş ve çıkış arasındaki ilişkiyi anlatan bir denklem elde etmek için kullanabiliriz. Özellikle Ohm yasasından, direnç boyunca $i(t)$ akımı direnç boyunca gerilim düşmesi ile orantılıdır (orantılılık sabiti 1/R ile); örneğin,

$$i(t) = \frac{v_s(t) - v_c(t)}{R} \qquad i(t) = C\frac{dv_c(t)}{dt} \qquad \boxed{\frac{dv_c(t)}{dt} + \frac{1}{RC}v_c(t) = \frac{1}{RC}v_s(t)}$$



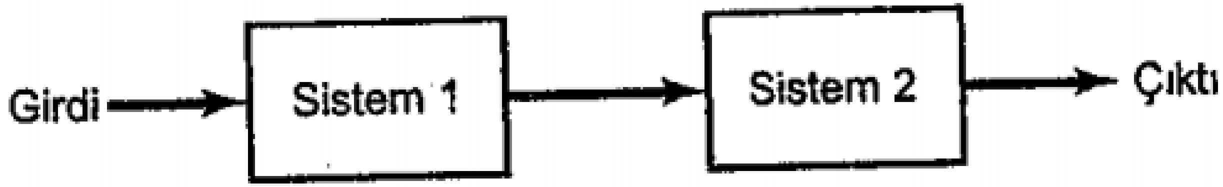
Şekil 1.1 Kaynak gerilimi v_s ve kondansatör gerilimi v_c ile basit bir RC devresi.

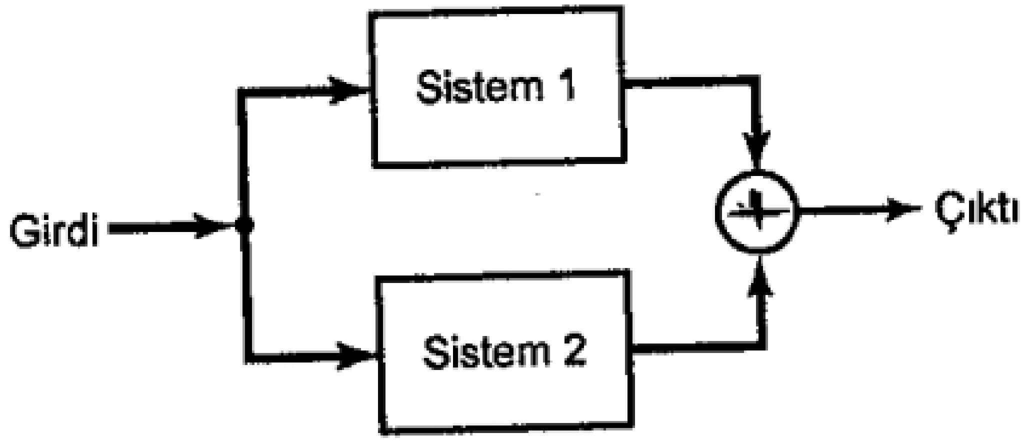
EXAMPLE 1.12 MOVING-AVERAGE SYSTEM Consider a discrete-time system whose output signal $y[n]$ is the average of the three most recent values of the input signal $x[n]$; that is,

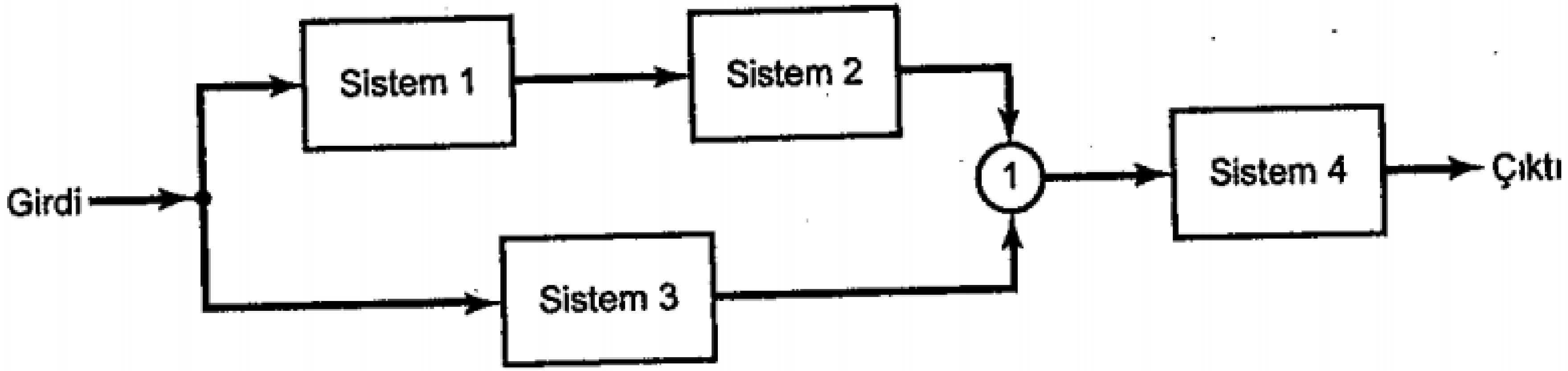
$$y[n] = \frac{1}{3}(x[n] + x[n-1] + x[n-2]).$$

Such a system is referred to as a *moving-average system*, for two reasons. First, $y[n]$ is the average of the sample values $x[n]$, $x[n-1]$, and $x[n-2]$. Second, the value of $y[n]$ changes as n moves along the discrete-time axis. Formulate the operator H for this system; hence, develop a block diagram representation for it.

Sistemlerin Ara Bağlantıları







TEMEL SİSTEM ÖZELLİKLERİ

Bellekli ve Belleksiz Sistemler

Bir sistem eğer belirli bir zamanda bağımsız değişkenin her değeri için aynı zamandaki çıkışa bağlı ise belleksiz denir. Örneğin,

$$y[n] = (2x[n] - x^2[n])^2$$

$$y(t) = Rx(t).$$

Bellekli bir ayrık zamanlı sistem örneği bir akümülatör veya toplayıcıdır; ikinci bir örnek ise bir gecikme 'dir:

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

$$y[n] = x[n-1].$$

Bir kapasitör bellekli bir sürekli zamanlı sisteme bir örnektir; çünkü eğer giriş akım olarak alınır ve çıkış gerilim olarak alınırsa C'nin kapasitans olduğu ;

$$y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)d\tau . \qquad (1.94)$$

Geri Dönüştürülebilirlik ve Ters Sistemler

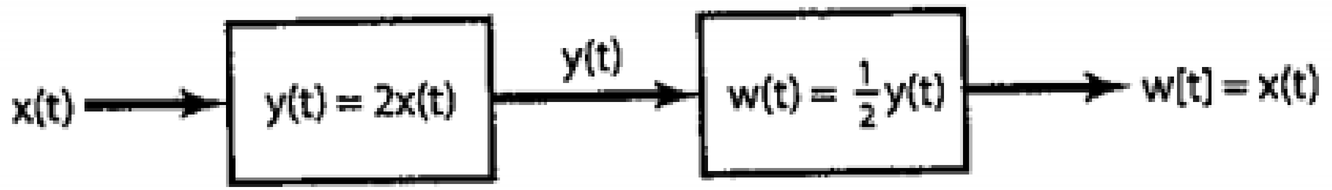
Bir sistem eğer farklı girişler farklı çıkışlara neden oluyor ise geri dönüştürülebilir denir. Şekil 1.45(a)'da ayrık zamanlı durum için gösterildiği gibi, eğer sistem geri dönüştürülebilir ise, orijinal bir sistem ile basamaklandığında ilk sistemin girişi x[n]'e eşit bir w[n] çıkışı üreten bir ters sistem mevcuttur. Böylece, Şekil 1.45(a)'da seri bağlantı, birim sistem için olan ile aynı tüm giriş-çıkış ilişkisine sahiptir.

Geri dönüştürülebilir bir sürekli zamanlı sistem örneği şöyledir:

$$y(t) = 2x(t), \tag{1.97}$$

ters sistem için ise şöyledir;

$$w(t) = \frac{1}{2}y(t). \tag{1.98}$$



Geri dönüştürülemez sistemlerin örnekleri şöyledir;

$$y[n] = 0, \tag{1.100}$$

yani, herhangi bir giriş dizisi için sıfır çıkışını üreten sistemdir ve çıkış bilgisinden girişin işaretini belirleyemediğimiz bir durumdur:

$$y(t) = x^2(t). \tag{1.101}$$

Nedensellik

Bir sistem, eğer herhangi bir andaki çıkış sadece o andaki ve önceki zamanlardaki çıkışların değerine bağlı ise nedenseldir. Bunun gibi bir sistem çoğunlukla nedensel olarak adlandırılır; çünkü sistem çıkışı girişlerin gelecek değerlerini tahmin etmez.

Örnek 1.12

Bir sistemin nedenselliğini kontrol ederken, giriş çıkış ilişkisine dikkatlice bakmak önemlidir. Bunun yaparken içerilen bazı konuları göstermek için iki özel sistemin nedenselliğini kontrol edeceğiz.

İlk sistem şöyle tanımlanmıştır:

$$y[n] = x[-n]. \tag{1.105}$$

$y[n_0]$ çıkışının n_0 pozitif zamanında sadece, negatif olan ve bu nedenle n_0'ın geçmişinde olan $(-n_0)$ zamanında $x[-n_0]$ giriş sinyalinin değerine bağlı olduğuna dikkat edelim. Bu noktada verilen sistemin nedensel olduğu sonucuna varmayı düşünebiliriz. Ancak, her zaman için giriş-çıkış ilişkisini kontrol için dikkat etmek gerekir. Özel olarak, $n < 0$ için, örneğin $n = -4$ için, $y[-4] = x[4]$ olduğunu görürüz; böylece bu andaki çıkış girişin bir gelecek değerine bağlıdır. Bu nedenle sistem nedensel değildir.

Ayrıca sistemin tanımında kullanılan herhangi diğer fonksiyonun girişinin etkilerini dikkatlice ayırt etmek önemlidir. Örneğin; aşağıdaki sistemi ele alalım;

$$y(t) = x(t)\cos(t + 1). \tag{1.106}$$

Bu nedenle, sadece $x(t)$ girişinin o andaki değeri $y(t)$ çıkışının o andaki değerini etkiler ve bu sistemin nedensel olduğu sonucuna varırız (ve aslında belleksiz).

Kararlılık

A system is said to be *bounded-input, bounded-output (BIBO) stable* if and only if every bounded input results in a bounded output. The output of such a system does not diverge if the input does not diverge.

To put the condition for BIBO stability on a formal basis, consider a continuous-time system whose input–output relation is as described in Eq. (1.78). The operator H is BIBO stable if the output signal $y(t)$ satisfies the condition

$$\boxed{|y(t)| \leq M_y < \infty \quad \text{for all } t} \tag{1.80}$$

whenever the input signals $x(t)$ satisfy the condition

$$\boxed{|x(t)| \leq M_x < \infty \quad \text{for all } t.} \tag{1.81}$$

Both M_x and M_y represent some finite positive numbers. We may describe the condition for the BIBO stability of a discrete-time system in a similar manner.

Example 1.13 Moving-Average System (continued)

Using the given input–output relation

$$y[n] = \frac{1}{3}(x[n] + x[n-1] + x[n-2]),$$

Solution: Assume that

$$|x[n]| < M_x < \infty \quad \text{for all } n.$$

Using the given input–output relation

$$y[n] = \frac{1}{3}(x[n] + x[n-1] + x[n-2]),$$

we may write

$$\begin{aligned} |y[n]| &= \frac{1}{3}|x[n] + x[n-1] + x[n-2]| \\ &\le \frac{1}{3}(|x[n]| + |x[n-1]| + |x[n-2]|) \\ &\le \frac{1}{3}(M_x + M_x + M_x) \\ &= M_x. \end{aligned}$$

Hence, the absolute value of the output signal $y[n]$ is always less than the maximum absolute value of the input signal $x[n]$ for all n, which shows that the moving-average system is stable. ■

EXAMPLE 1.14 UNSTABLE SYSTEM Consider a discrete-time system whose input–output relation is defined by

$$y[n] = r^n x[n],$$

where $r > 1$. Show that this system is unstable.

Solution: Assume that the input signal $x[n]$ satisfies the condition

$$|x[n]| \leq M_x < \infty \quad \text{for all } n.$$

We then find that

$$\begin{aligned} |y[n]| &= |r^n x[n]| \\ &= |r^n| \cdot |x[n]|. \end{aligned}$$

With $r > 1$, the multiplying factor r^n diverges for increasing n. Accordingly, the condition that the input signal is bounded is not sufficient to guarantee a bounded output signal, so the system is unstable. To prove stability, we need to establish that all bounded inputs produce a bounded output. ■

▶ **Problem 1.26** The input-output relation of a discrete-time system is described by

$$y[n] = \sum_{k=0}^{\infty} \rho^k x[n-k].$$

Show that the system is BIBO unstable if $|\rho| \geq 1$. ◀

Time Invariance

A system is said to be *time invariant* if a time delay or time advance of the input signal leads to an identical time shift in the output signal. This implies that a time-invariant system responds identically no matter when the input signal is applied. Put another way, the characteristics of a time-invariant system do not change with time. Otherwise, the system is said to be *time variant*.

$x(t - t_0) \rightarrow y(t - t_0)$ $x[k - k_0] \rightarrow y[k - k_0]$

Example 2.4

Consider two CT systems represented mathematically by the following input–output relationship:

(i) system I $y(t) = \sin(x(t));$ (2.42)

(ii) system II $y(t) = t\sin(x(t)).$ (2.43)

Determine if systems (i) and (ii) are time-invariant.

Solution

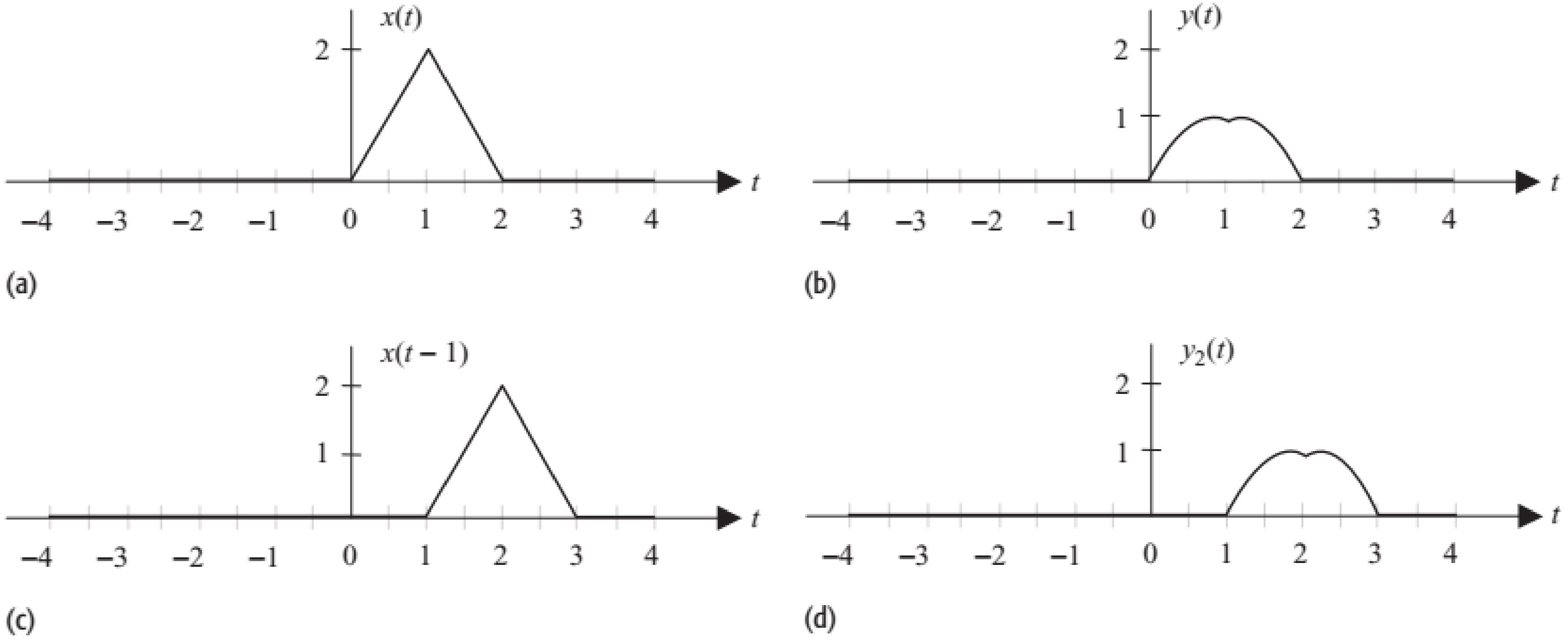
(i) From Eq. (2.42), it follows that:

$$x(t) \rightarrow \sin(x(t)) = y(t)$$

and

$$x(t - t_0) \rightarrow \sin(x(t - t_0)) = y(t - t_0).$$

Since $\sin[x(t - t_0)] = y(t - t_0)$, system I is time-invariant. We demonstrate the time-invariance property of system I graphically in Fig. 2.13, where a time-shifted version $x(t - 1)$ of input $x(t)$ produces an equal shift of one time unit in the original output $y(t)$ obtained from $x(t)$.



(ii) From Eq. (2.43), it follows that:

$$x(t) \rightarrow t \sin(x(t)) = y(t).$$

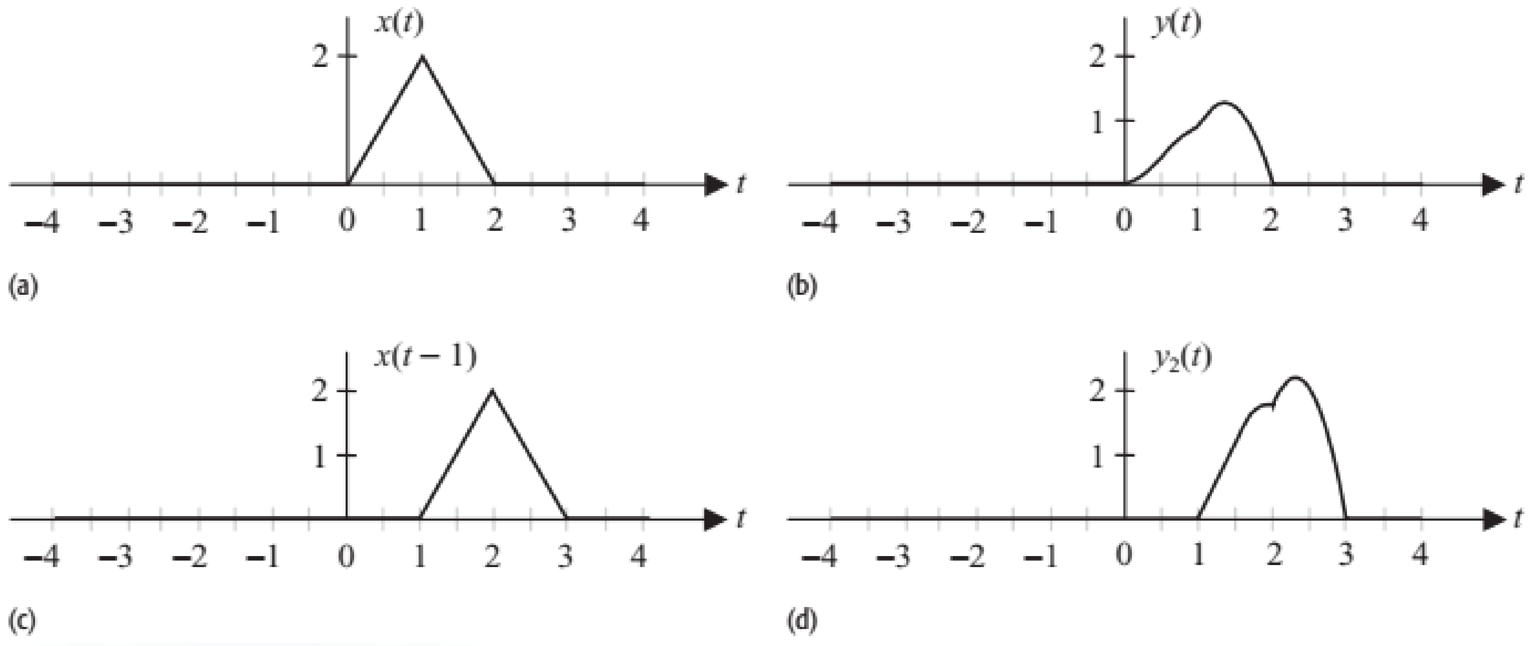
If the time-shifted signal $x(t - t_0)$ is applied at the input of Eq. (2.43), the new output is given by

$$x(t - t_0) \rightarrow t \sin(x(t - t_0)).$$

The shifted output $y(t - t_0)$ is given by

$$y(t - t_0) = (t - t_0) \sin(x(t - t_0)).$$

Since $t \sin[x(t - t_0)] \neq y(t - t_0)$, system II is not time-invariant. The time-



LINEARITY

A system is said to be *linear* in terms of the system input (excitation) $x(t)$ and the system output (response) $y(t)$ if it satisfies the following two properties of superposition and homogeneity:

1. *Superposition.* Consider a system that is initially at rest. Let the system be subjected to an input $x(t) = x_1(t)$, producing an output $y(t) = y_1(t)$. Suppose next that the same system is subjected to a different input $x(t) = x_2(t)$, producing a corresponding output $y(t) = y_2(t)$. Then for the system to be linear, it is necessary that the composite input $x(t) = x_1(t) + x_2(t)$ produce the corresponding output $y(t) = y_1(t) + y_2(t)$. What we have described here is a statement of the *principle of superposition* in its simplest form.
2. *Homogeneity.* Consider again a system that is initially at rest, and suppose an input $x(t)$ results in an output $y(t)$. Then the system is said to exhibit the property of homogeneity if, whenever the input $x(t)$ is scaled by a constant factor a, the output $y(t)$ is scaled by exactly the same constant factor a.

Sürekli zamanlı: $ax_1(t) + bx_2(t) \rightarrow ay_1(t) + by_2(t)$,

Ayrık zamanlı: $ax_1[n] + bx_2[n] \rightarrow ay_1[n] + by_2[n]$.

Örnek 1.17

$x(t)$ girişi ve $y(t)$ çıkışı

$$y(t) = tx(t)$$

ile ilişkili bir S sistemini ele alalım. S'nin doğrusal olup olmadığını belirlemek için, iki rasgele giriş $x_1(t)$ ve $x_2(t)$'yi ele alalım.

$$x_1(t) \to y_1(t) = tx_1(t)$$

$$x_2(t) \to y_2(t) = tx_2(t)$$

$x_3(t)$, $x_1(t)$ ve $x_2(t)$'nin doğrusal bir kombinasyonu olsun. Yani, a ve b'nin rasgele nicelikler olduğu,

$$x_3(t) = ax_1(t) + bx_2(t)\ .$$

Eğer $x_3(t)$ S'ye giriş ise, bu durumda karşılık gelen çıkış şöyle ifade edilebilir:

$$\begin{aligned} y_3(t) &= tx_3(t) \\ &= t(ax_1(t) + bx_2(t)) \\ &= atx_1(t) + btx_2(t) \\ &= ay_1(t) + by_2(t) \end{aligned}$$

Sistemin doğrusal olduğu sonucuna varırız.

Örnek 1.18

Yukarıdaki örneğin doğrusallık kontrol yöntemini, $x(t)$ girişi ve $y(t)$ çıkışı

$$y(t) = x^2(t)$$

ile ilişkilendirilmiş başka bir sistem S'e uygulayalım. Önceki örnekte olduğu gibi $x_1(t)$, $x_2(t)$ ve $x_3(t)$'yi tanımlarsak:

$$x_1(t) \rightarrow y_1(t) = x_1^2(t)$$
$$x_2(t) \rightarrow y_2(t) = x_2^2(t)$$

ve

$$\begin{aligned} x_3(t) \rightarrow y_3(t) &= x_3^2(t) \\ &= (ax_1(t) + bx_2(t))^2 \\ &= a^2x_1^2(t) + b^2x_2^2(t) + 2abx_1(t)x_2(t) \\ &= a^2y_1(t) + b^2y_2(t) + 2abx_1(t)x_2(t) \end{aligned}$$

Açıkça, $x_1(t)$, $x_2(t)$, a ve b'yi, $y_3(t)$ $ay_1(t) + by_2(t)$ ile aynı olmayacak şekilde belirleyebiliriz. Örneğin, eğer $x_1(t) = 1$, $x_2(t) = 0$, $a = 2$ ve $b = 0$ ise bu durumda $y_3(t) = (2x_1(t))^2$ 'dir ancak $2y_1(t) = (2x_1(t))^2 = 2$ 'dir. s sisteminin doğrusal olmadığı sonucuna varırız.

Örnek 1.20

$$y[n] = 2x[n] + 3 \tag{1.132}$$

denklemini ele alalım. Birçok şekilde doğrulanabileceği gibi, bu sistem doğrusal değildir. Örneğin, sistem toplanırlık özelliğine uymaz: eğer $x_1[n] = 2$ ve $x_2[n] = 3$ ise bu durumda :

$$x_1[n] \rightarrow y_1[n] = 2x_1[n] + 3 = 7\,, \tag{1.133}$$

$$x_2[n] \rightarrow y_2[n] = 2x_2[n] + 3 = 9\,. \tag{1.134}$$

Ancak, $x_3[n] = x_1[n] + x_2[n]$ 'e tepki şöyledir:

$$y_3[n] = 2[x_1[n] + x_2[n]] + 3 = 13 \tag{1.135}$$

ve $y_1[n] + y_2[n] = 16$ 'e eşit değildir. Alternatif olarak, $x[n] = 0$ ise $y[n]=3$ olduğundan, sistemin (1.125) denkleminde verilen doğrusal sistemlerin "sıfır giriş-sıfır çıkış" özelliğine uymadığını görürüz.

Example 2.1

Consider the CT systems with the following input–output relationships:

(a) differentiator $$y(t) = \frac{\mathrm{d}x(t)}{\mathrm{d}t}; \tag{2.33}$$

(b) exponential amplifier $$x(t) \to \mathrm{e}^{x(t)}; \tag{2.34}$$

(c) amplifier $$y(t) = 3x(t); \tag{2.35}$$

(d) amplifier with additive bias $$y(t) = 3x(t) + 5. \tag{2.36}$$

Determine whether the CT systems are linear.

Solution

(a) From Eq. (2.33), it follows that

$$x_1(t) \to \frac{\mathrm{d}x_1(t)}{\mathrm{d}t} = y_1(t)$$

and

$$x_2(t) \to \frac{\mathrm{d}x_2(t)}{\mathrm{d}t} = y_2(t),$$

which yields

$$\alpha x_1(t) + \beta_1 x_2(t) \to \frac{\mathrm{d}}{\mathrm{d}t}\{\alpha x_1(t) + \beta_1 x_2(t)\} = \alpha\frac{\mathrm{d}x_1(t)}{\mathrm{d}t} + \beta\frac{\mathrm{d}x_2(t)}{\mathrm{d}t}.$$

Since

$$\alpha\frac{\mathrm{d}x_1(t)}{\mathrm{d}t} + \beta\frac{\mathrm{d}x_2(t)}{\mathrm{d}t} = \alpha y_1(t) + \beta y_2(t),$$

(b) From Eq. (2.34), it follows that

$$x_1(t) \to \mathrm{e}^{x_1(t)} = y_1(t)$$

and

$$x_2(t) \to \mathrm{e}^{x_2(t)} = y_2(t),$$

giving

$$\alpha x_1(t) + \beta x_2(t) \to \mathrm{e}^{\alpha x_1(t)+\beta x_2(t)}.$$

Since

$$\mathrm{e}^{\alpha x_1(t)+\beta x_2(t)} = \mathrm{e}^{\alpha x_1(t)} \cdot \mathrm{e}^{\beta x_2(t)} = [y_1(t)]^{\alpha} + [y_2(t)]^{\beta} \neq \alpha y_1(t) + \beta y_2(t).$$

the exponential amplifier represented by Eq. (2.34) is not a linear system.

(c) From (2.35), it follows that

$$x_1(t) \rightarrow 3x_1(t) = y_1(t)$$

and

$$x_2(t) \rightarrow 3x_2(t) = y_2(t),$$

giving

$$\alpha x_1(t) + \beta x_2(t) \rightarrow 3\{\alpha x_1(t) + \beta x_2(t)\} = 3\alpha x_1(t) + 3\beta x_2(t)$$
$$= \alpha y_1(t) + \beta y_2(t).$$

Therefore, the amplifier of Eq. (2.35) is a linear system.

An alternative approach to check if a system is non-linear is to apply the zero-input, zero-output property. For system (b), if $x(t) = 0$, then $y(t) = 1$. System (b) does not satisfy the zero-input, zero-output property, hence system (b) is non-linear. Likewise, for system (d), if $x(t) = 0$ then $y(t) = 5$. Therefore, system (d) is not a linear system.

If a system does not satisfy the zero-input, zero-output property, we can safely classify the system as a non-linear system. On the other hand, if it satisfies the zero-input, zero-output property, it can be linear or non-linear. Satisfying the zero-input, zero-output property is not a sufficient condition to prove the linearity of a system. A CT system $y(t) = x^2(t)$ is clearly a non-linear system, yet it satisfies the zero-input, zero-output property.

(d) From Eq. (2.36), we can write

$$x_1(t) \rightarrow 3x_1(t) + 5 = y_1(t)$$

and

$$x_2(t) \rightarrow 3x_2(t) + 5 = y_2(t),$$

giving

$$\alpha x_1(t) + \beta x_2(t) \rightarrow 3[\alpha x_1(t) + \beta x_2(t)] + 5.$$

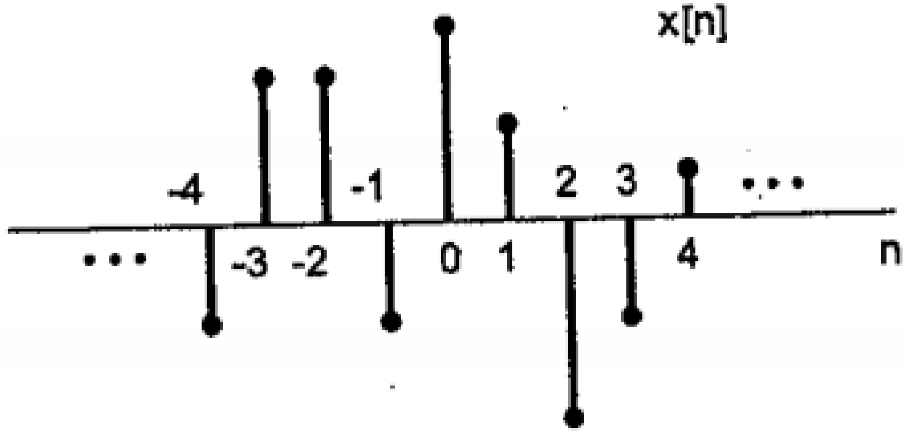
Since

$$3[\alpha x_1(t) + \beta x_2(t)] + 5 = \alpha y_1(t) + \beta y_2(t) - 5,$$

Time-Domain Representations of Linear Time-Invariant Systems

Ayrık Zamanlı Sinyallerin Dürtüler Bakımından İfadesi

Ayrık zamanlı birim dürtünün herhangi bir ayrık zamanlı sinyalin oluşturulmasında nasıl kullanılacağını gözde canlandırırken anahtar fikir, ayrık zamanlı sinyali dürtülerin tek başına bir sonucu olarak düşünmektir. Bu sezgisel portrenin nasıl matematiksel bir ifadeye dönüşebileceğini görmek için, Şekil 2.1(a)'da gösterilen $x[n]$ sinyalini ele alalım. Bu şeklin kalan kısımlarında, her dürtüde ölçeklemenin birim örneğin oluştuğu anda belirli $x[n]$ değerine eşit olduğu beş zamanda kaydırılmış ölçeklenmiş birim dürtü dizileri gösterilmiştir. Örneğin;

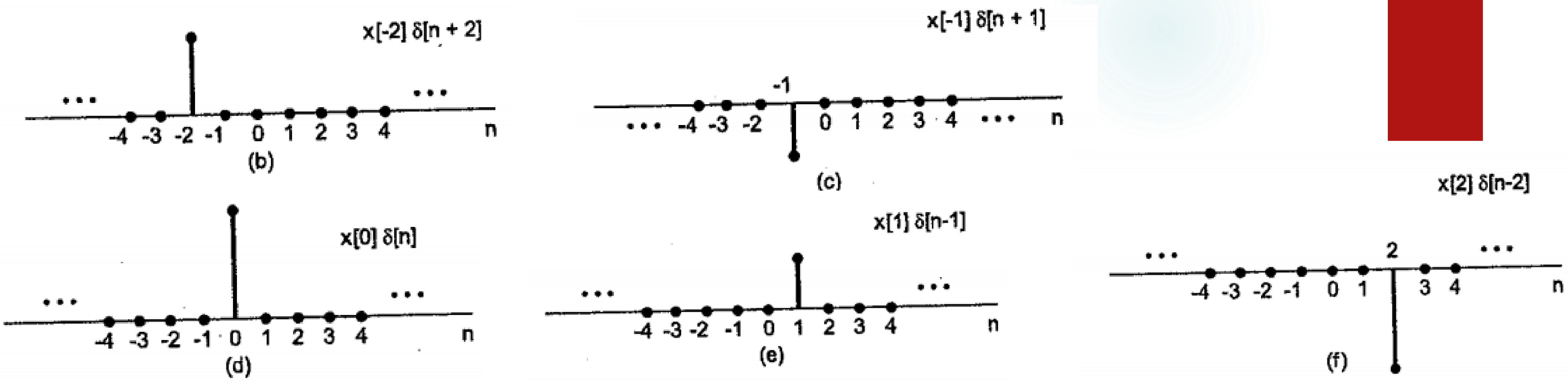


(a)

$$x[-1]\delta[n+1] = \begin{cases} x[-1], & n=-1 \\ 0, & n \neq -1 \end{cases},$$

$$x[0]\delta[n] = \begin{cases} x[0], & n=0 \\ 0, & n \neq 0 \end{cases},$$

$$x[1]\delta[n-1] = \begin{cases} x[1], & n=1 \\ 0, & n \neq 1 \end{cases}$$



Bu nedenle, şekildeki beş dizinin toplamı $-2 \leq n \leq 2$ için $x[n]$'dir. Daha genel olarak, ekstra kaydırma, ölçeklenmiş dürtüler dâhil edilerek, şunu yazabiliriz:

$$x[n] = \ldots + x[-3]\delta[n+3] + x[-2]\delta[n+2] + x[-1]\delta n+1] + x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + x[3]\delta[n-3] + \ldots \quad (2.1)$$

$$\boxed{x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]}$$

Bu, rasgele bir dizinin, lineer kombinasyonda ağırlıkların $x[k]$ olduğu $\delta[n-k]$ kaydırılmış birim dürtülerin lineer bir kombinasyonu olarak ifadesine karşılık gelir. Örnek olarak, $x[n] = u[n]$ birim basamağını ele alalım. Bu durumda, k <0 için $u[k] = 0$ olduğundan ve $k \geq 0$ için $u[k] = 1$ olduğundan, (2.2) denklemi bölüm 1.4'de elde edilen [bkz. (1.67) denklemi] ifadeye eşit olan şu hale gelir:

$$u[n] = \sum_{k=0}^{+\infty} \delta[n-k]$$

Ayrık Zamanlı Birim Dürtü Tepkisi ve LTI Sistemlerin Evrişim Toplamı Gösterimi

Let the operator H denote the system to which the input $x[n]$ is applied. Then, using Eq. (2.1) to represent the input $x[n]$ to the system results in the output

$$y[n] = H\{x[n]\}$$

$$= H\left\{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right\}.$$

$$x[n] = \cdots + x[-2]\delta[n+2] + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + \cdots.$$

Now we use the linearity property to interchange the system operator H with the summation and obtain

$$y[n] = \sum_{k=-\infty}^{\infty} H\{x[k]\delta[n-k]\}.$$

Since n is the time index, the quantity $x[k]$ is a constant with respect to the system operator H. Using linearity again, we interchange H with $x[k]$ to obtain

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]H\{\delta[n-k]\}. \tag{2.2}$$

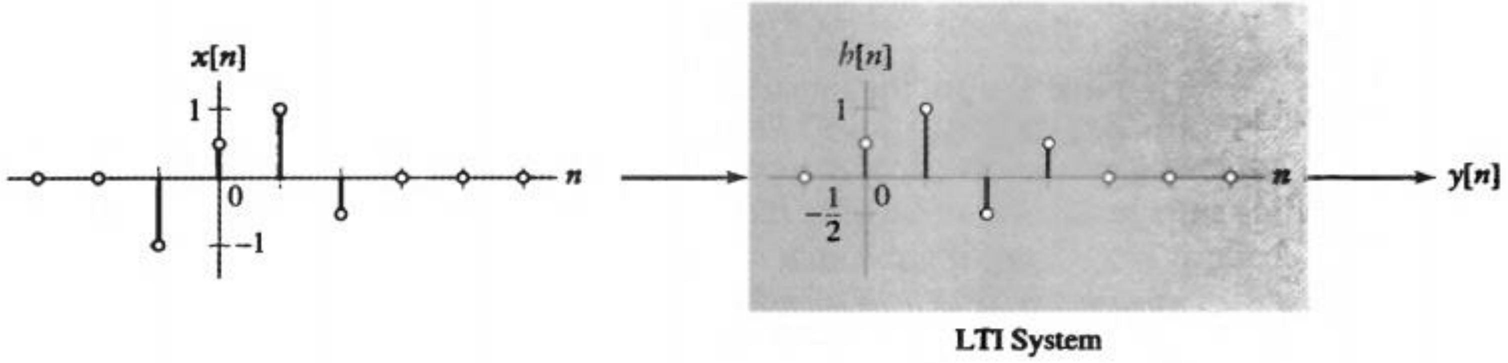
If we further assume that the system is time invariant, then a time shift in the input results in a time shift in the output. This relationship implies that the output due to a time-shifted impulse is a time-shifted version of the output due to an impulse; that is,

$$H\{\delta[n-k]\} = h[n-k], \tag{2.3}$$

$$\boxed{y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k].}$$

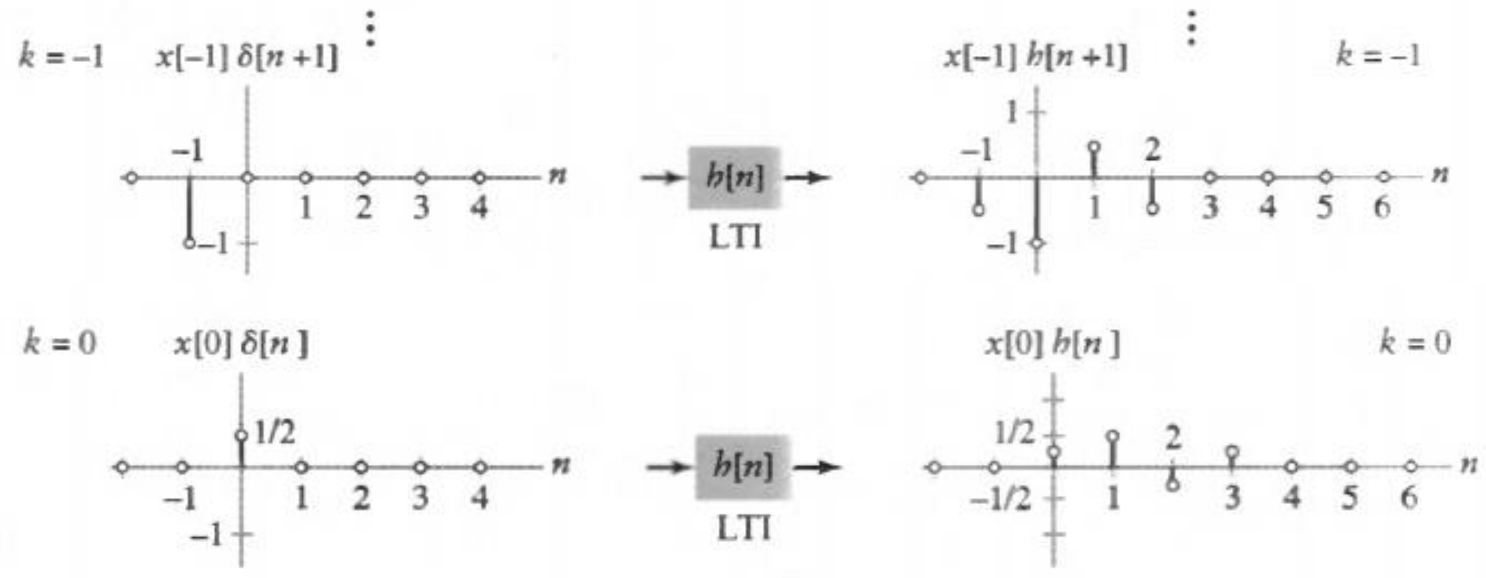
Thus, the output of an LTI system is given by a weighted sum of time-shifted impulse responses. This is a direct consequence of expressing the input as a weighted sum of time-shifted impulse basis functions. The sum in Eq. (2.4) is termed the *convolution sum* and is denoted by the symbol $*$; that is,

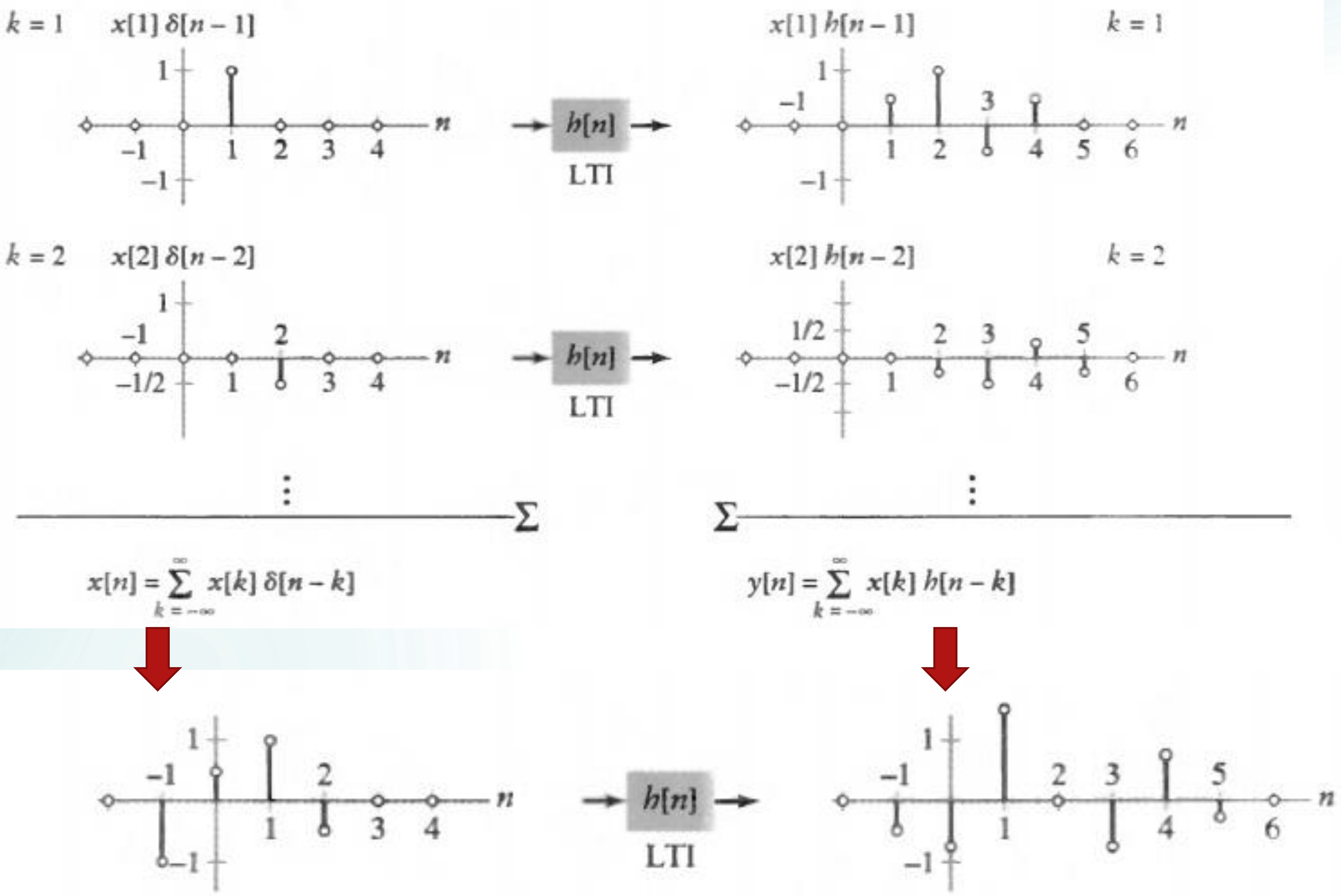
$$\boxed{x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k].}$$



(a)

FIGURE 2.2 Illustration of the convolution sum. (a) LTI system with impulse response $h[n]$ and input $x[n]$, producing the output $y[n]$ to be determined.





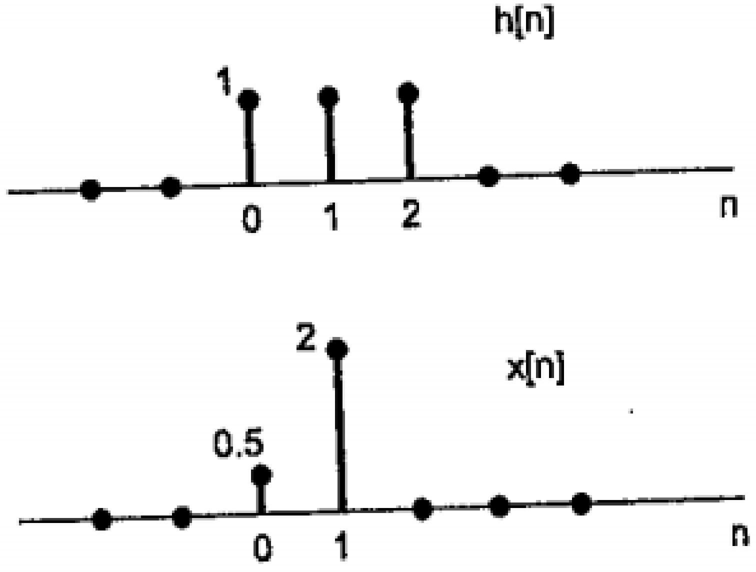
k = 1
x[1] δ[n – 1]
h[n]
LTI
x[1] h[n – 1]
k = 1
k = 2
x[2] δ[n – 2]
h[n]
LTI
x[2] h[n – 2]
k = 2
Σ
Σ
$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$
$y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$
h[n]
LTI

Örnek 2.1

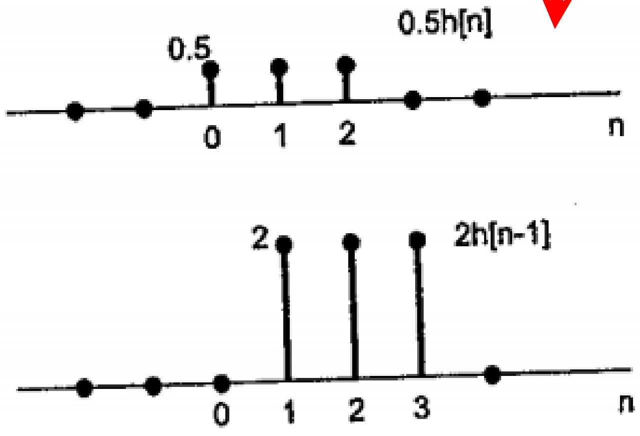
Şekil 2.3(a)'da gösterildiği gibi $x[n]$ girişli ve $h[n]$ dürtü tepkili bir LTI sistemini ele alalım. Bu durum için sadece $x[0]$ ve $x[1]$ sıfırdan farklı olduğu için, (2.6) denklemi şu ifadeye sadeleşir:

$$y[n] = x[0]h[n-0] + x[1]h[n-1] = 0.5h[n] + 2h[n-1]. \tag{2.8}$$

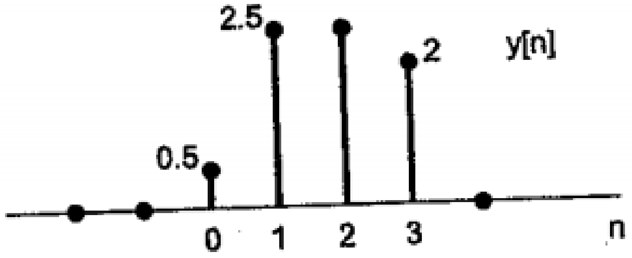
$0.5h[n]$ ve $2h[n-1]$ dizileri, $y[n]$'in üretimi ile ilgili bindirme için gerekli dürtü yankısıdır. Bu yankılar şekil 2.3(b)'de gösterilmiştir. n'in her değeri için iki yankıyı toplayarak, Şekil 2.3(c)'de gösterilen $y[n]$ elde edilir.



(a)



(b)



(c)

Example

$$y[n] = x[n] + \frac{1}{2}x[n-1].$$

a) h[n]=?

Letting $x[n] = \delta[n]$, we find that the impulse response is

$$h[n] = \begin{cases} 1, & n = 0 \\ \frac{1}{2}, & n = 1 \\ 0, & \text{otherwise} \end{cases}.$$

b) Determine the output of this system in response to the input

$$x[n] = \begin{cases} 2, & n = 0 \\ 4, & n = 1 \\ -2, & n = 2 \\ 0, & \text{otherwise} \end{cases}.$$

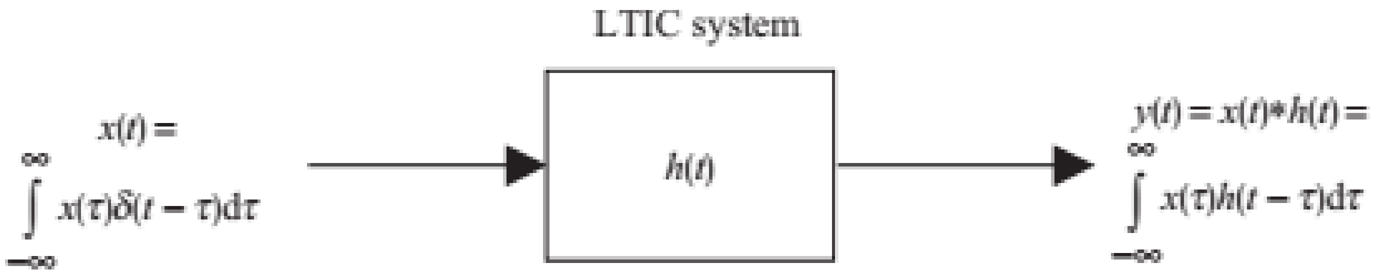
Solution: First, write $x[n]$ as the weighted sum of time-shifted impulses:

$$x[n] = 2\delta[n] + 4\delta[n-1] - 2\delta[n-2].$$

$$y[n] = 2h[n] + 4h[n-1] - 2h[n-2].$$

$$y[n] = \begin{cases} 0, & n < 0 \\ 2, & n = 0 \\ 5, & n = 1 \\ 0, & n = 2 \\ -1, & n = 3 \\ 0, & n \geq 4 \end{cases}.$$

The Convolution Integral



convolution of two functions $x(t)$ and $h(t)$ is defined as follows:

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\mathrm{d}\tau .$$

Example 3.6

Determine the output response of an LTIC system when the input signal is given by $x(t) = \exp(-t)u(t)$ and the impulse response is $h(t) = \exp(-2t)u(t)$.

Solution

Using Eq. (3.36), the output $y(t)$ of the LTIC system is given by

$$y(t) = \int_{-\infty}^{\infty} e^{-\tau}u(\tau)e^{-2(t-\tau)}u(t-\tau)d\tau,$$

which can be expressed as

$$y(t) = e^{-2t}\int_{0}^{\infty} e^{\tau}u(t-\tau)d\tau.$$

Expressed as a function of the independent variable τ, the unit step function is given by

$$u(t-\tau) = \begin{cases} 1 & \tau \le t \\ 0 & \tau > t. \end{cases}$$

Based on the value of t, we have the following two cases for the output $y(t)$.

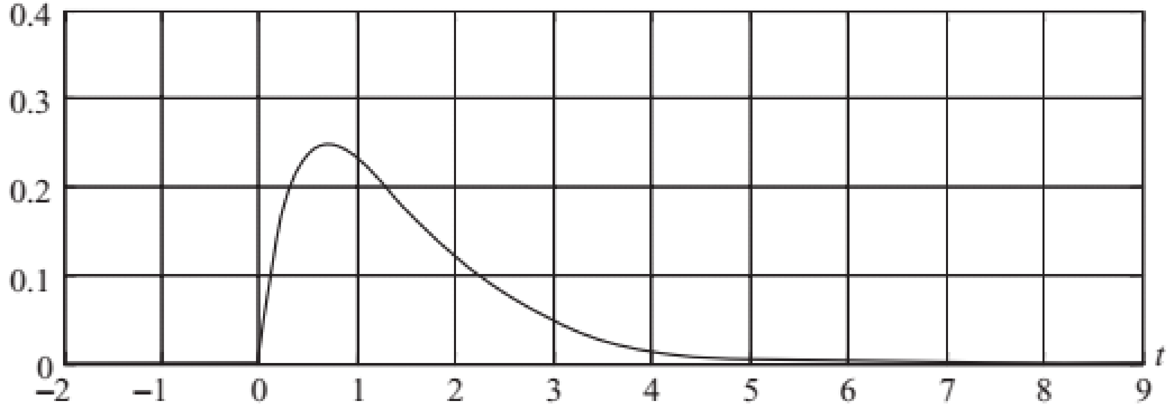
Case I For $t < 0$, the shifted unit step function $u(t - \tau) = 0$ within the limits of integration $[0, \infty]$. Therefore, $y(t) = 0$ for $t < 0$.

Case II For $t \geq 0$, the shifted unit step function $u(t - \tau)$ has two different values within the limits of integration $[0, \infty]$. For the range $[0, t]$, the unit step function $u(t - \tau) = 1$. Otherwise, for the range $[t, \infty]$, the unit step function is zero. The output $y(t)$ is therefore given by

$$y(t) = e^{-2t} \int_0^t e^{\tau}\, d\tau = e^{-2t}\left[e^{t} - 1\right] = e^{-t} - e^{-2t}, \quad \text{for } t > 0.$$

Combining cases I and II, the overall output $y(t)$ is given by

$$y(t) = (e^{-t} - e^{-2t})u(t).$$



Graphical method for evaluating the convolution integral

Box 3.1 Steps for graphical convolution

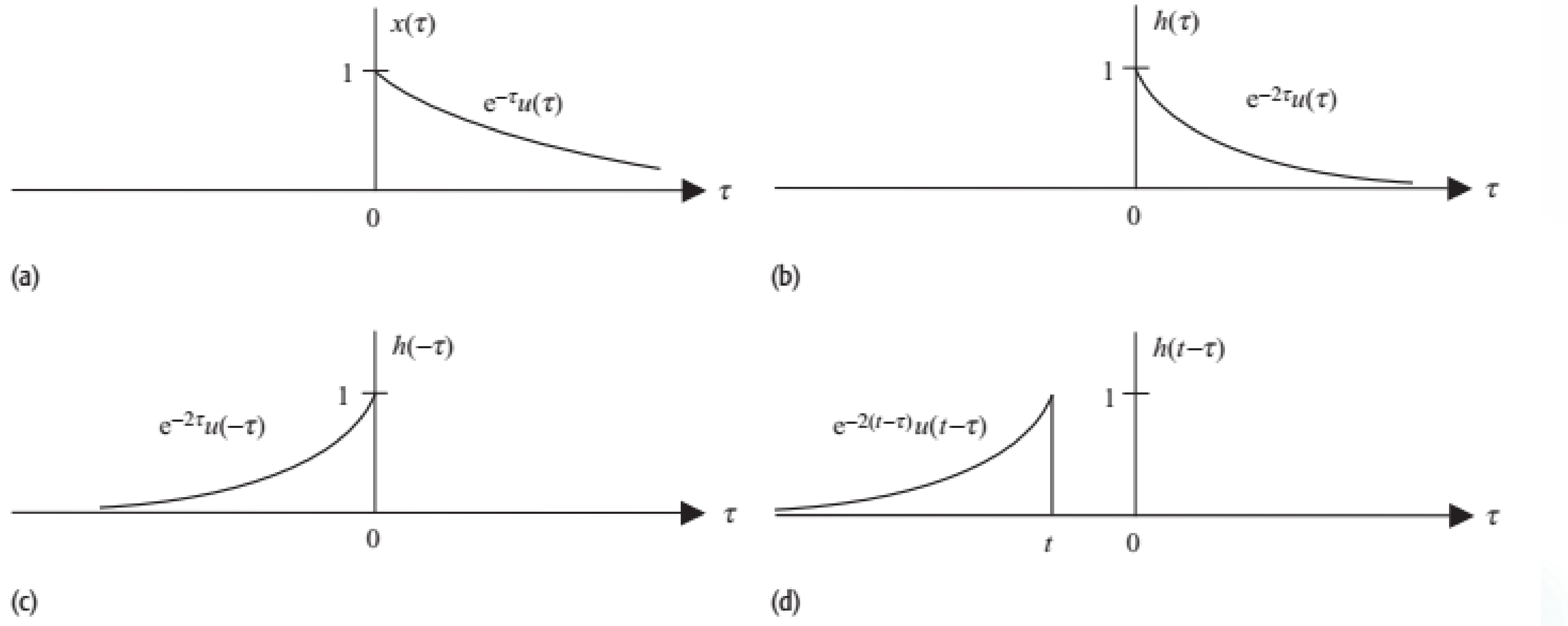
(1) Sketch the waveform for input $x(\tau)$ by changing the independent variable from t to τ and keep the waveform for $x(\tau)$ fixed during convolution.
(2) Sketch the waveform for the impulse response $h(\tau)$ by changing the independent variable from t to τ.
(3) Reflect $h(\tau)$ about the vertical axis to obtain the time-inverted impulse response $h(-\tau)$.
(4) Shift the time-inverted impulse function $h(-\tau)$ by a selected value of "t." The resulting function represents $h(t-\tau)$.
(5) Multiply function $x(\tau)$ by $h(t-\tau)$ and plot the product function $x(\tau)h(t-\tau)$.
(6) Calculate the total area under the product function $x(\tau)h(t-\tau)$ by integrating it over $\tau = [-\infty, \infty]$.
(7) Repeat steps 4–6 for different values of t to obtain $y(t)$ for all time, $-\infty \le t \le \infty$.

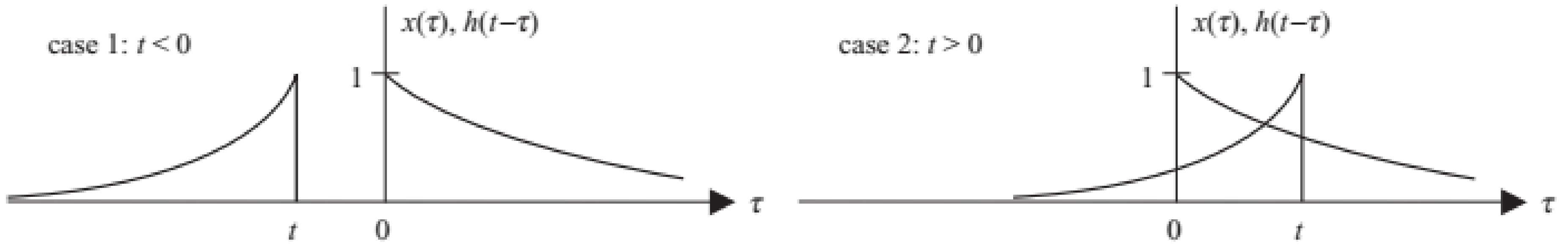
Example 3.7

Repeat Example 3.6 and determine the zero-state response of the system using the graphical convolution method.

Solution

Functions $x(\tau) = \exp(-\tau)u(\tau)$, $h(\tau) = \exp(-2\tau)u(\tau)$, and $h(-\tau) = \exp(-2\tau)u(-\tau)$ are plotted, respectively, in Figs. 3.7(a)–(c). The function $h(t-\tau) = h(-(\tau - t))$ is obtained by shifting $h(-\tau)$ by time t. We consider the following two cases of t.





(e) (f)

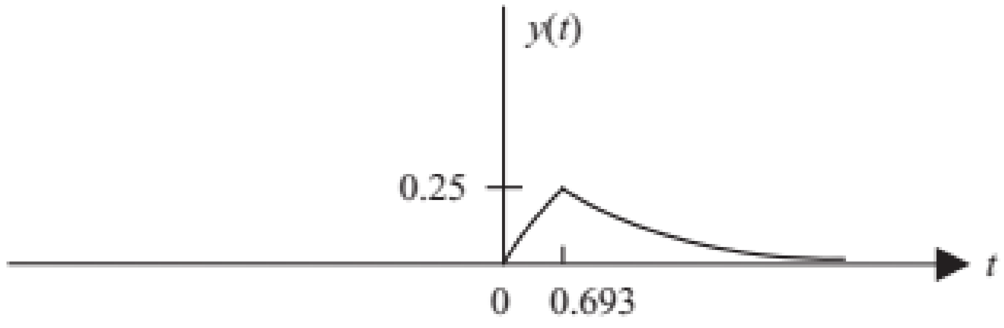
Case 1 For $t < 0$, the waveform $h(t-\tau)$ is on the left-hand side of the vertical axis. As is apparent in Fig. 3.7(e), waveforms for $h(t-\tau)$ and $x(\tau)$ do not overlap. In other words, $x(\tau)h(t-\tau) = 0$ for all τ, hence $y(t) = 0$.

Case 2 For $t \geq 0$, we see from Fig. 3.7(f) that the non-zero parts of $h(t-\tau)$ and $x(\tau)$ overlap over the duration $t = [0, t]$. Therefore,

$$y(t) = \int_0^t e^{-2t+\tau} d\tau = e^{-2t} \int_0^t e^{\tau} d\tau = e^{-2t}[e^t - 1] = e^{-t} - e^{-2t}.$$

Combining the two cases, we obtain

$$y(t) = \begin{cases} 0 & t < 0 \\ e^{-t} - e^{-2t} & t \geq 0, \end{cases}$$

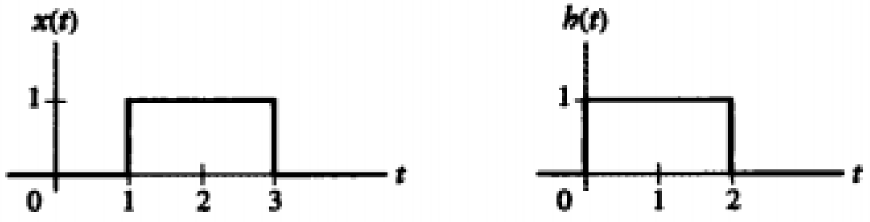


EXAMPLE 2.6 REFLECT-AND-SHIFT CONVOLUTION EVALUATION Evaluate the convolution integral for a system with input $x(t)$ and impulse response $h(t)$, respectively, given by

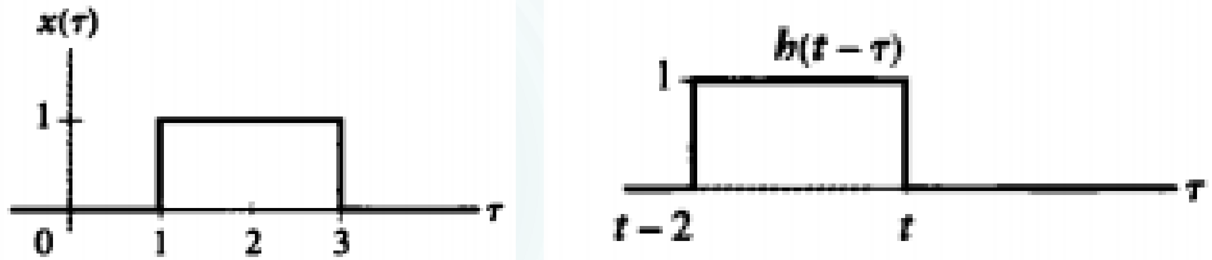
$$x(t) = u(t-1) - u(t-3)$$

and

$$h(t) = u(t) - u(t-2),$$



Solution:



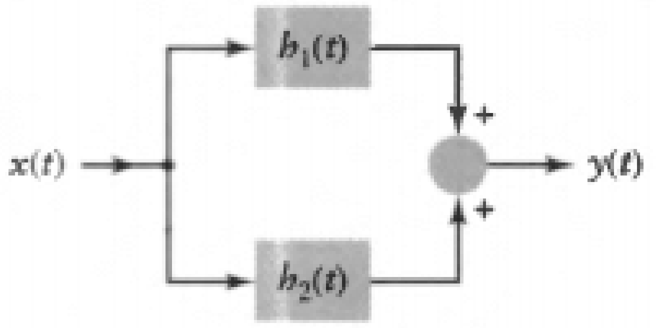
(a)

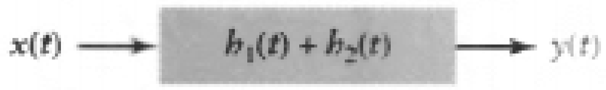
Combining the solutions for each interval of time shifts gives the output

$$y(t) = \begin{cases} 0, & t < 1 \\ t-1, & 1 \le t < 3 \\ 5-t, & 3 \le t < 5 \\ 0, & t \ge 5 \end{cases},$$

Interconnections of LTI Systems

Parallel Connection of LTI Systems





$$y(t) = y_1(t) + y_2(t)$$
$$= x(t) * h_1(t) + x(t) * h_2(t).$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h_1(t-\tau)\,d\tau + \int_{-\infty}^{\infty} x(\tau)h_2(t-\tau)\,d\tau.$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\{h_1(t-\tau) + h_2(t-\tau)\}\,d\tau$$
$$= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$
$$= x(t) * h(t),$$

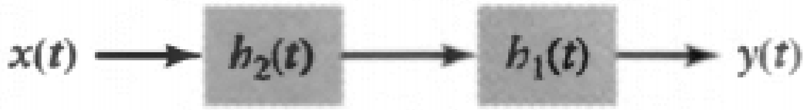
$$\boxed{x(t) * h_1(t) + x(t) * h_2(t) = x(t) * \{h_1(t) + h_2(t)\}.}$$

$$\boxed{x[n] * h_1[n] + x[n] * h_2[n] = x[n] * \{h_1[n] + h_2[n]\}.}$$

Cascade Connection of Systems

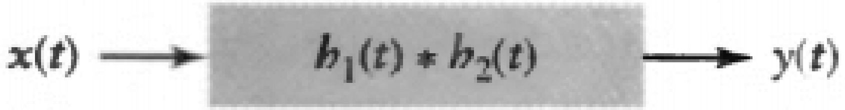
$$y(t) = z(t) * h_2(t),$$

$$y(t) = \int_{-\infty}^{\infty} z(\tau) h_2(t-\tau)\, d\tau.$$



$$z(\tau) = x(\tau) * h_1(\tau)$$

$$= \int_{-\infty}^{\infty} x(\nu) h_1(\tau-\nu)\, d\nu,$$



$$y(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\nu) h_1(\tau-\nu) h_2(t-\tau)\, d\nu d\tau.$$

$$\boxed{\{x(t) * h_1(t)\} * h_2(t) = x(t) * \{h_1(t) * h_2(t)\}.}$$

$$\boxed{\{x[n] * h_1[n]\} * h_2[n] = x[n] * \{h_1[n] * h_2[n]\},}$$

$$\boxed{h_1(t) * h_2(t) = h_2(t) * h_1(t).}$$

$$\boxed{h_1[n] * h_2[n] = h_2[n] * h_1[n].}$$

TABLE 2.1 ***Interconnection Properties for LTI Systems.***

Property	*Continuous-time system*	*Discrete-time system*
Distributive	$x(t) * h_1(t) + x(t) * h_2(t) = x(t) * \{h_1(t) + h_2(t)\}$	$x[n] * h_1[n] + x[n] * h_2[n] = x[n] * \{h_1[n] + h_2[n]\}$
Associative	$\{x(t) * h_1(t)\} * h_2(t) = x(t) * \{h_1(t) * h_2(t)\}$	$\{x[n] * h_1[n]\} * h_2[n] = x[n] * \{h_1[n] * h_2[n]\}$
Commutative	$h_1(t) * h_2(t) = h_2(t) * h_1(t)$	$h_1[n] * h_2[n] = h_2[n] * h_1[n]$

Shift property If $x_1(t) * x_2(t) = g(t)$ then

$$x_1(t - T_1) * x_2(t - T_2) = g(t - T_1 - T_2),$$

Convolution with impulse function

$$x(t) * \delta(t - t_0) = x(t - t_0).$$

Convolution with unit step function

$$x(t) * u(t) = \int_{-\infty}^{\infty} x(\tau)u(t - \tau)\mathrm{d}\tau = \int_{-\infty}^{t} x(\tau)\mathrm{d}\tau.$$

Relations between LTI System Properties and the Impulse Response

Memoryless LTIC systems

a memoryless LTIC system typically has an input–output relationship of the form

$$y(t) = kx(t),$$

where k is a constant. Substituting $x(t) = \delta(t)$, the impulse response $h(t)$ of a memoryless system can be obtained as follows:

$$h(t) = k\delta(t). \tag{3.43}$$

An LTIC system will be memoryless if and only if its impulse response $h(t) = 0$ for $t \neq 0$.

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k].$$

$$\Rightarrow \quad y[n] = \cdots + h[-2]x[n+2] + h[-1]x[n+1] + h[0]x[n] + h[1]x[n-1] + h[2]x[n-2] + \cdots$$

For this system to be memoryless, $y[n]$ must depend only on $x[n]$ and therefore cannot depend on $x[n-k]$ for $k \neq 0$. Hence, every term in Eq. (2.27) must be zero, except $h[0]x[n]$. This condition implies that $h[k] = 0$ for $k \neq 0$; thus, a discrete-time LTI system is memoryless if and only if

$$\boxed{h[k] = c\delta[k],}$$

Causal LTI Systems

The output of a causal LTI system depends only on past or present values of the input. Again, we write the convolution sum as

$$y[n] = \cdots + h[-2]x[n+2] + h[-1]x[n+1] + h[0]x[n] + h[1]x[n-1] + h[2]x[n-2] + \cdots.$$

We see that past and present values of the input, $x[n], x[n-1], x[n-2], \ldots$, are associated with indices $k \geq 0$ in the impulse response $h[k]$, while future values of the input, $x[n+1], x[n+2], \ldots$, are associated with indices $k < 0$. In order, then, for $y[n]$ to depend only on past or present values of the input, we require that $h[k] = 0$ for $k < 0$. Hence, for a discrete-time causal LTI system,

$$\boxed{h[k] = 0 \quad \text{for} \quad k < 0,}$$

and the convolution sum takes the new form

$$y[n] = \sum_{k=0}^{\infty} h[k]x[n-k].$$

A causal continuous-time LTI system has an impulse response that satisfies the condition

$$\boxed{h(\tau) = 0 \quad \text{for} \quad \tau < 0.} \quad \Longrightarrow \quad y(t) = \int_0^{\infty} h(\tau)x(t-\tau)\,d\tau.$$

Stable LTI Systems

A CT system is BIBO stable if an arbitrary bounded input signal produces a bounded output signal. Consider a bounded signal $x(t)$ with $|x(t)| < B_x$ for all t, applied as input to an LTIC system with impulse response $h(t)$. The magnitude of output $y(t)$ is given by

$$|y(t)| = \left| \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\mathrm{d}\tau \right|.$$

Using the Schwartz inequality, we can say that the output is bounded within the range

$$|y(t)| \leq \int_{-\infty}^{\infty} |h(\tau)||x(t-\tau)|\mathrm{d}\tau.$$

Since $x(t)$ is bounded, $|x(t)| < B_x$, therefore the above inequality reduces to

$$|y(t)| \leq B_x \int_{-\infty}^{\infty} |h(\tau)|\mathrm{d}\tau.$$

$$\int_{-\infty}^{\infty} |h(t)|\mathrm{d}t < \infty,$$

$$\boxed{\sum_{k=-\infty}^{\infty} |h[k]| < \infty.}$$

Example 3.10

Determine if systems with the following impulse responses:

(i) $h(t) = \delta(t) - \delta(t-2)$,

(ii) $h(t) = 2\,\mathrm{rect}(t/2)$,

(iii) $h(t) = 2\exp(-4t)u(t)$,

(iv) $h(t) = [1 - \exp(-4t)]u(t)$,

Solution

System (i)

Memoryless property. Since $h(t) \neq 0$ for $t \neq 0$, system (i) is not memoryless.

Causality property. Since $h(t) = 0$ for $t < 0$, system (i) is causal.

Stability property. To verify if system (i) is stable, we compute the following integral:

$$\int_{-\infty}^{\infty} |h(t)|\mathrm{d}t = \int_{-\infty}^{\infty} |\delta(t) - \delta(t-2)|\mathrm{d}t$$

$$\leq \int_{-\infty}^{\infty} |\delta(t)|\mathrm{d}t + \int_{-\infty}^{\infty} |\delta(t-2)|\mathrm{d}t = 2 < \infty,$$

which shows that system (i) is stable.

System (ii)

Memoryless property. Since $h(t) \neq 0$ for $t \neq 0$, system (ii) is not memoryless.

Causality property. Since $h(t) \neq 0$ for $t < 0$, system (ii) is not causal.

Stability property. To verify if system (ii) is stable, we compute the following integral:

$$\int_{-\infty}^{\infty} |h(t)|\mathrm{d}t = \int_{-1}^{1} 2\,\mathrm{d}t = 4 < \infty,$$

which shows that system (ii) is stable.

System (iii)

Memoryless property. Since $h(t) \neq 0$ for $t \neq 0$, system (iii) is not memoryless. The memory of system (iii) is infinite, as the output at any time instant depends on the values of the input taken over the entire past.

Causality property. Since $h(t) = 0$ for $t < 0$, system (iii) is causal.

Stability property. To verify that system (iii) is stable, we solve the following integral:

$$\int_{-\infty}^{\infty} |h(t)|\mathrm{d}t = \int_{0}^{\infty} 2\mathrm{e}^{-4t}\mathrm{d}t = -0.5 \times [\mathrm{e}^{-4t}]_0^{\infty} = 0.5 < \infty,$$

which shows that system (iii) is stable.

System (iv)

Memoryless property. Since $h(t) \neq 0$ for $t \neq 0$, system (iv) is not memoryless.

Causality property. Since $h(t) = 0$ for $t < 0$, system (iv) is causal.

Stability property. To verify that system (iv) is stable, we solve the following integral:

$$\int_{-\infty}^{\infty} |h(t)|\mathrm{d}t = \int_{0}^{\infty} (1 - \mathrm{e}^{-4t})\mathrm{d}t = [t - 0.25\mathrm{e}^{-4t}]_0^{\infty} = \infty,$$

which shows that system (iv) is not stable.

Invertible LTIC systems

Consider an LTIC system with impulse response $h(t)$. The output $y_1(t)$ of the system for an input signal $x(t)$ is given by $y_1(t) = x(t) * h(t)$. For the system to be invertible, we cascade a second system with impulse response $h_i(t)$ in series with the original system. The output of the second system is given by

$$y_2(t) = y_1(t) * h_i(t).$$

For the second system to be an inverse of the original system, output $y_2(t)$ should be the same as $x(t)$. Substituting $y_1(t) = x(t) * h(t)$ in the above expression results in the following condition for invertibility:

$$x(t) = [x(t) * h(t)] * h_i(t) = x(t) * [h(t) * h_i(t)].$$

The above equation is true if and only if

$$h(t) * h_i(t) = \delta(t). \tag{3.45}$$

TABLE 2.2 ***Properties of the Impulse Response Representation for LTI Systems.***

Property	*Continuous-time system*	*Discrete-time system*
Memoryless	$h(t) = c\delta(t)$	$h[n] = c\delta[n]$
Causal	$h(t) = 0 \quad \text{for} \quad t < 0$	$h[n] = 0 \quad \text{for} \quad n < 0$
Stability	$\int_{-\infty}^{\infty} \lvert h(t) \rvert \, dt < \infty$	$\sum_{n=-\infty}^{\infty} \lvert h[n] \rvert < \infty$
Invertibility	$h(t) * h^{\text{inv}}(t) = \delta(t)$	$h[n] * h^{\text{inv}}[n] = \delta[n]$

2.9 *Differential and Difference Equation Representations of LTI Systems*

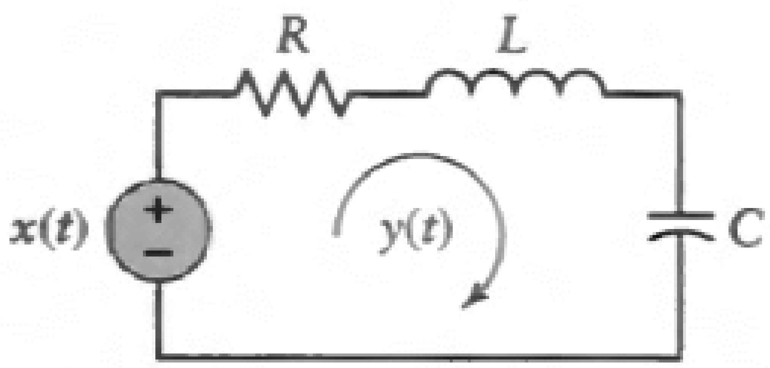
The general form of a linear constant-coefficient differential equation is

$$\boxed{\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t),}$$

where the a_k and the b_k are constant coefficients of the system, $x(t)$ is the input applied to the system, and $y(t)$ is the resulting output. A linear constant-coefficient difference equation has a similar form, with the derivatives replaced by delayed values of the input $x[n]$ and output $y[n]$:

$$\boxed{\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k].}$$

The *order* of the differential or difference equation is (N, M), representing the number of energy storage devices in the system. Often, $N \geq M$, and the order is described using only N.



$$Ry(t) + L\frac{d}{dt}y(t) + \frac{1}{C}\int_{-\infty}^{t} y(\tau)\, d\tau = x(t).$$

Differentiating both sides of this equation with respect to t results in

$$\frac{1}{C}y(t) + R\frac{d}{dt}y(t) + L\frac{d^2}{dt^2}y(t) = \frac{d}{dt}x(t).$$

This differential equation describes the relationship between the current $y(t)$ and the voltage $x(t)$ in the circuit. In this example, the order is $N = 2$, and we note that the circuit contains two energy storage devices: a capacitor and an inductor.

An example of a second-order difference equation is

$$y[n] + y[n-1] + \frac{1}{4}y[n-2] = x[n] + 2x[n-1], \qquad (2.37)$$

which may represent the relationship between the input and output signals of a system that processes data in a computer. Here, the order is $N = 2$, because the difference equation involves $y[n-2]$, implying a maximum memory of 2 in the system output. Memory in a discrete-time system is analogous to energy storage in a continuous-time system.

Difference equations are easily rearranged to obtain recursive formulas for computing the current output of the system from the input signal and past outputs. We rewrite Eq. (2.36) so that $y[n]$ is alone on the left-hand side:

$$\boxed{y[n] = \frac{1}{a_0}\sum_{k=0}^{M} b_k x[n-k] - \frac{1}{a_0}\sum_{k=1}^{N} a_k y[n-k].}$$

This equation indicates how to obtain $y[n]$ from the present and past values of the input and the past values of the output. Such equations are often used to implement discrete-time systems in a computer. Consider computing $y[n]$ for $n \geq 0$ from $x[n]$ for the second-order difference equation (2.37) , rewritten in the form

$$y[n] = x[n] + 2x[n-1] - y[n-1] - \frac{1}{4}y[n-2]. \qquad (2.38)$$

Beginning with $n = 0$, we may determine the output by evaluating the sequence of equations

$$y[0] = x[0] + 2x[-1] - y[-1] - \frac{1}{4}y[-2], \tag{2.39}$$

$$y[1] = x[1] + 2x[0] - y[0] - \frac{1}{4}y[-1], \tag{2.40}$$

$$y[2] = x[2] + 2x[1] - y[1] - \frac{1}{4}y[0],$$

$$y[3] = x[3] + 2x[2] - y[2] - \frac{1}{4}y[1],$$

In each equation, the current output is computed from the input and past values of the output. In order to begin this process at time $n = 0$, we must know the two most recent past values of the output, namely, $y[-1]$ and $y[-2]$. These values are known as *initial conditions*.

the initial conditions are $y[-1] = 1$ and $y[-2] = -2$.

$$y[n] = x[n] + 2x[n-1] - y[n-1] - \frac{1}{4}y[n-2]. \rightarrow$$

$$y[0] = 1 + 2 \times 0 - 1 - \frac{1}{4} \times (-2) = \frac{1}{2}.$$

$$y[1] = \frac{1}{2} + 2 \times 1 - \frac{1}{2} - \frac{1}{4} \times (1) = 1\frac{3}{4}.$$

► **Problem 2.14** Write a differential equation describing the relationship between the input voltage $x(t)$ and current $y(t)$ through the inductor in Fig. 2.29.

Answer:

$$Ry(t) + L\frac{d}{dt}y(t) = x(t).$$ ◄

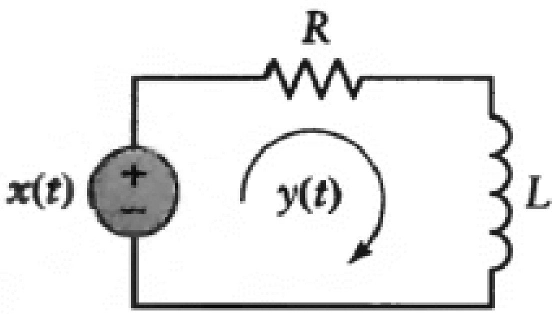


FIGURE 2.29 RL circuit.

► **Problem 2.15** Calculate $y[n], n = 0, 1, 2, 3$ for the first-order recursive system

$$y[n] - (1/2)y[n-1] = x[n]$$

if the input is $x[n] = u[n]$ and the initial condition is $y[-1] = -2$.

Answer:

$$y[0] = 0, \quad y[1] = 1, \quad y[2] = 3/2, \quad y[3] = 7/4.$$ ◄

Solving Differential and Difference Equations

THE HOMOGENEOUS SOLUTION

The homogeneous form of a differential or difference equation is obtained by setting all terms involving the input to zero. Hence, for a continuous-time system, $y^{(h)}(t)$ is the solution of the homogeneous equation

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y^{(h)}(t) = 0.$$

The homogeneous solution for a continuous-time system is of the form

$$\boxed{y^{(h)}(t) = \sum_{i=1}^{N} c_i e^{r_i t},} \tag{2.41}$$

where the r_i are the N roots of the system's *characteristic equation*

$$\boxed{\sum_{k=0}^{N} a_k r^k = 0.} \tag{2.42}$$

Substitution of Eq. (2.41) into the homogeneous equation establishes the fact that $y^{(h)}(t)$ is a solution for any set of constants c_i.

In discrete time, the solution of the homogeneous equation

$$\sum_{k=0}^{N} a_k y^{(h)}[n-k] = 0$$

is

$$\boxed{y^{(h)}[n] = \sum_{i=1}^{N} c_i r_i^n,} \tag{2.43}$$

where the r_i are the N roots of the discrete-time system's characteristic equation

$$\boxed{\sum_{k=0}^{N} a_k r^{N-k} = 0.} \tag{2.44}$$

EXAMPLE 2.17 RC CIRCUIT: HOMOGENEOUS SOLUTION The RC circuit depicted in Fig. 2.30 is described by the differential equation

$$y(t) + RC\frac{d}{dt}y(t) = x(t).$$

Determine the homogeneous solution of this equation.

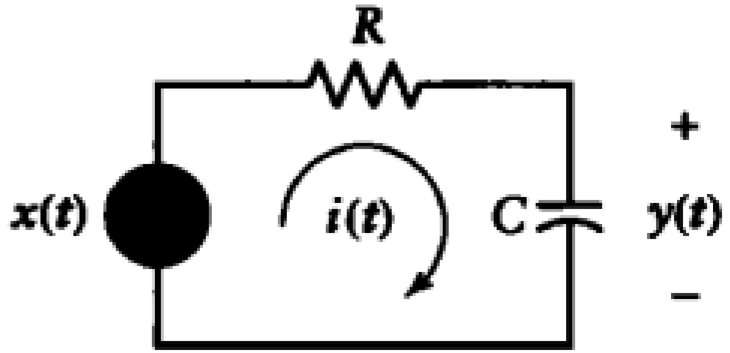


FIGURE 2.30 RC circuit.

Solution: The homogeneous equation is

$$y(t) + RC\frac{d}{dt}y(t) = 0.$$

The solution is given by Eq. (2.41), using $N = 1$ to obtain

$$y^{(h)}(t) = c_1 e^{r_1 t}\,\mathrm{V},$$

where r_1 is the root of the characteristic equation

$$1 + RCr_1 = 0.$$

Hence, $r_1 = -\frac{1}{RC}$, and the homogeneous solution for this system is

$$y^{(h)}(t) = c_1 e^{-\frac{t}{RC}}\,\mathrm{V}. \qquad \blacksquare$$

► **Problem 2.16** Determine the homogeneous solution for the systems described by the following differential or difference equations:

(a)

$$\frac{d^2}{dt^2}y(t) + 5\frac{d}{dt}y(t) + 6y(t) = 2x(t) + \frac{d}{dt}x(t)$$

(b)

$$\frac{d^2}{dt^2}y(t) + 3\frac{d}{dt}y(t) + 2y(t) = x(t) + \frac{d}{dt}x(t)$$

(c)

$$y[n] - (9/16)y[n-2] = x[n-1]$$

(d)

$$y[n] + (1/4)y[n-2] = x[n] + 2x[n-2]$$

Answers:

(a)

$$y^{(h)}(t) = c_1e^{-3t} + c_2e^{-2t}$$

(b)

$$y^{(h)}(t) = c_1e^{-t} + c_2e^{-2t}$$

(c)

$$y^{(h)}[n] = c_1(3/4)^n + c_2(-3/4)^n$$

(d)

$$y^{(h)}[n] = c_1(1/2e^{j\pi/2})^n + c_2(1/2e^{-j\pi/2})^n$$

TABLE 2.3 ***Form of Particular Solutions Corresponding to Commonly Used Inputs.***

Continuous Time		Discrete Time	
Input	*Particular Solution*	*Input*	*Particular Solution*
1	c	1	c
t	$c_1 t + c_2$	n	$c_1 n + c_2$
e^{-at}	ce^{-at}	α^n	$c\alpha^n$
$\cos(\omega t + \phi)$	$c_1 \cos(\omega t) + c_2 \sin(\omega t)$	$\cos(\Omega n + \phi)$	$c_1 \cos(\Omega n) + c_2 \sin(\Omega n)$

EXAMPLE 2.20 *RC* CIRCUIT (CONTINUED): PARTICULAR SOLUTION Consider the *RC* circuit of Example 2.17 and depicted in Fig. 2.30. Find a particular solution for this system with an input $x(t) = \cos(\omega_0 t)$.

Solution: From Example 2.17, the differential equation describing the system is

$$y(t) + RC\frac{d}{dt}y(t) = x(t).$$

We assume a particular solution of the form $y^{(p)}(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t)$. Replacing $y(t)$ in the differential equation by $y^{(p)}(t)$ and $x(t)$ by $\cos(\omega_0 t)$ gives

$$c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) - RC\omega_0 c_1\sin(\omega_0 t) + RC\omega_0 c_2\cos(\omega_0 t) = \cos(\omega_0 t).$$

The coefficients c_1 and c_2 are obtained by separately equating the coefficients of $\cos(\omega_0 t)$ and $\sin(\omega_0 t)$. This gives the following system of two equations in two unknowns:

$$c_1 + RC\omega_0 c_2 = 1;$$
$$-RC\omega_0 c_1 + c_2 = 0.$$

Solving these equations for c_1 and c_2 gives

$$c_1 = \frac{1}{1 + (RC\omega_0)^2}$$

and

$$c_2 = \frac{RC\omega_0}{1 + (RC\omega_0)^2}.$$

Hence, the particular solution is

$$y^{(p)}(t) = \frac{1}{1 + (RC\omega_0)^2}\cos(\omega_0 t) + \frac{RC\omega_0}{1 + (RC\omega_0)^2}\sin(\omega_0 t)\text{ V}. \qquad \blacksquare$$

THE COMPLETE SOLUTION

Procedure 2.3: Solving a Differential or Difference Equation

1. Find the form of the homogeneous solution $y^{(h)}$ from the roots of the characteristic equation.
2. Find a particular solution $y^{(p)}$ by assuming that it is of the same form as the input, yet is independent of all terms in the homogeneous solution.
3. Determine the coefficients in the homogeneous solution so that the complete solution $y = y^{(p)} + y^{(h)}$ satisfies the initial conditions.

► **Problem** **Aşağıda verilen sistemin homojen çözümünü ve $x(t)=e^{-t}$ girdisi için özel çözümünü hesaplayınız**

$$\frac{d^2}{dt^2}y(t) + 5\frac{d}{dt}y(t) + 6y(t) = 2x(t) + \frac{d}{dt}x(t)$$